Mathematical Physics I: Assignment 5

HRI Graduate School
August - December 2010

23 November 2010
To be returned on 30 November 2010

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. The Hermite differential equation is given by

$$y''(x) - 2xy'(x) + 2\alpha y(x) = 0.$$

(i) Find the solution of the above equation using series substitution about $x = 0$ and obtain two linearly independent solutions. Obtain the first three non-vanishing terms in the both the series.
(ii) Investigate the convergence of the two series solutions. Under what condition(s) does the solution(s) reduce to polynomials?
(iii) Write the differential equation in the self-adjoint form. What is the corresponding weight function $w(x)$?
(iv) Write a possible orthogonality relation for solutions of Hermite equation for different values of α.

$[4 + 2 + 2 + 2 = 10]$

2. A quantum mechanical analysis of the **Stark effect** (parabolic coordinates) leads to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}u}{\mathrm{d}\xi}\right) + \left(\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right)u = 0$$

where α is a separation constant, E is the total energy, and F is a constant, where Fz is the potential energy added to the system by the introduction of an electric field.

Using the larger root of the indicial equation, develop a power-series solution about $\xi = 0$. Evaluate the first three coefficients in terms of a_0.

[4]

3. The radial Schrödinger wave equation has the form

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + V(r)\right\}\psi(r) = E\psi(r)$$

Assume that the potential energy $V(r)$ may be expanded about the origin as

$$V(r) = \frac{b_{-1}}{r} + b_0 + b_1 r + \ldots$$

(i) Find the first two terms of the regular solution.
(ii) Show that the irregular solution diverges at the origin as r^{-l}.

$[3 + 2 = 5]$

4. Find the two linearly independent solutions for the equation

$$y'' + \frac{1-\alpha^2}{4x^2}y = 0$$

What happens when $\alpha = 0$?

[4]

5. Consider the equation for a damped harmonic oscillator:

$$y''(x) + 2\gamma y'(x) + \omega_0^2 y(x) = F(x); \qquad x > 0$$

where γ, ω_0 are constants. Assume $\gamma < \omega_0$.
(i) Find the solutions $y_1(x), y_2(x)$ of the corresponding homogeneous differential equation. What is the Wronskian?
(ii) Find the Green's function for the boundary condition $y(0) = y'(0) = 0$ using the method involving the Wronskian.
(iii) Find the Green's function using the method of Fourier transform. Make sure you show the results of the complex contour integration explicitly.

[3 + 4 + 6 = 13]

6. Find the Green's function $G(x, s)$ for the following operators with the boundary conditions as mentioned:
(i)

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}; 0 < x < 1 \quad G(0, s) = G'(1, s) = 0$$

(ii)

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2; \quad G(-\infty, s) = G(\infty, s) = 0$$

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right); 0 < x < 1; \quad \lim_{x\to 0} |G(x, s)| < \infty, G(1, s) = 0$$

(iv)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) - \frac{m^2}{x}; 0 < x < 1; \quad \lim_{x\to 0} |G(x, s)| < \infty, G(1, s) = 0$$

[6 × 4 = 24]