Assignment 0

09 August 2010

The questions in this assignment are based on standard courses at B.Sc./M.Sc. level. You may look up textbooks and/or consult friends for solving the problems, but make sure you understand the solutions.

You need *not* submit this assignment. However, if you find any of these questions nontrivial/difficult, please talk to your tutor for this course.

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1. Prove that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = (AB)^2 - (\mathbf{A} \cdot \mathbf{B})^2$

2. Let $g(\mathbf{r})$ be a scalar function and $\mathbf{f}(\mathbf{r})$ a vector function. Suppose $\mathbf{f}(\mathbf{r})$ is *not* irrotational but the product $g(\mathbf{r})\mathbf{f}(\mathbf{r})$ is irrotational, then show that
$$\mathbf{f} \cdot \nabla \times \mathbf{f} = 0.$$

3. If $\mathbf{r}$ is a vector from the origin to the point (x, y, z), calculate the following:
$$\text{(i) } \nabla \times \mathbf{r}, \qquad \text{(ii) } \nabla \times \left(\frac{\mathbf{r}}{r^3}\right), \qquad \text{(iii) } \nabla \left(\frac{1}{r}\right)$$

4. Prove that $\nabla \times (\phi \nabla \phi) = 0$ where ϕ is a scalar function.

5. Show that
$$\frac{1}{3} \oiint_S \mathbf{r} \cdot \mathrm{d}\sigma = V,$$
where V is the volume enclosed by the surface $S = \partial V$.

6. Consider a point charge at the origin producing an electric field :
$$\mathbf{E} = \frac{q\,\mathbf{r}}{4\pi\,\epsilon\,r^3}\,.$$
Now, using the expression for ∇ in spherical polar coordinates, calculate the value of
$$\nabla \cdot \frac{\mathbf{r}}{r^3}.$$
By Gauss law, we must have
$$\int_S \mathbf{E} \,.\, \mathrm{d}\mathbf{s} = \int_V \nabla \cdot \mathbf{E}\,\mathrm{d}V = \frac{q}{\epsilon}\,,$$
for any closed surface (say, a sphere) enclosing the origin. If you take $\mathbf{E}$ corresponding to the point charge and put in the expression for Gauss law, do you get the desired result?

7. Let $\delta(x)$ be the Dirac delta function. Show that
$$\delta[(x - x_1)(x - x_2)] = \frac{\delta(x - x_1) + \delta(x - x_2)}{|x_1 - x_2|}$$

8. Show that for any matrix A, $\det \mathsf{A}^{-1} = (\det \mathsf{A})^{-1}$

9. The Pauli spin matrices are
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
Show that
(i) $[\sigma_i, \sigma_j] = 2\mathrm{i}\varepsilon_{ijk}\sigma_k$ where ε_{ijk} is the Levi-Civita symbol
(ii) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}1_2$ where 1_2 is the 2×2 identity matrix
(iii) $\sigma_i\sigma_j = \delta_{ij}1_2 + \mathrm{i}\sum_k \varepsilon_{ijk}\sigma_k$
(iv) $(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}1_2 + \mathrm{i}\sigma \cdot (\mathbf{a} \times \mathbf{b})$, where $\sigma = \sigma_1\hat{\mathbf{x}} + \sigma_2\hat{\mathbf{y}} + \sigma_3\hat{\mathbf{z}}$ and $\mathbf{a}, \mathbf{b}$ are ordinary vectors

10. Let there be a system of point masses:
$$\begin{aligned} m_1 &= 1 \quad \text{at} \quad (1, 1, -2) \\ m_2 &= 2 \quad \text{at} \quad (-1, -1, 0) \\ m_3 &= 1 \quad \text{at} \quad (1, 1, 2) \end{aligned}$$

(i) The inertia matrix, which relates the angular velocity to the angular momentum, is defined as

$$I^{ij} = \sum_a m_a (r_a^2 \delta^{ij} - x_a^i x_a^j); \quad i,j = 1,2,3$$

where the sum is over all discrete point masses. Compute the inertia matrix for the above system.
(ii) Diagonalize it by obtaining the eigenvalues and the orthonormal eigenvectors.

11. Calculate the real and imaginary part of the quantity

$$\left(\frac{\mathrm{i}a - 1}{\mathrm{i}a + 1}\right)^{ib}$$

where a, b are real.

12. Complex numbers $a + ib$ may be represented by 2×2 matrices:

$$a + \mathrm{i}b \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(i) Is this representation valid? Check for addition and multiplication.
(ii) What is the matrix corresponding to $(a + \mathrm{i}b)^{-1}$?
(iii) What do the determinant and transpose of the matrices represent?

13. We know that the logarithm (Ln) of a complex number (z) is multivalued. The *principal value of* $\mathrm{Ln}\, z$ is denoted as $\ln\, z$ and is defined as the value of $\mathrm{Ln}\, z$ obtained by restricting the argument of z to lie in the range $-\pi < \theta \leq \pi$. Hence evaluate $\mathrm{Ln}(-\mathrm{i})$ and find out $\ln(-\mathrm{i})$.

14. Obtain the Taylor series expansions (upto $\mathcal{O}(x^5)$) of the following functions about $x = 0$:
(i)

$$\frac{\sinh x}{x},$$

(ii)

$$\ln \cos x,$$

(iii)

$$\mathrm{e}^x \sec x,$$

(iv)

$$\ln(x + \sqrt{1 + x^2})$$

15. Check the convergence of the following infinite series and evaluate them if convergent:
(i)

$$\sum_{n=0}^{\infty} \frac{n!}{10^n},$$

(ii)

$$\sum_{n=0}^{\infty} \frac{\mathrm{e}^n}{n^5},$$

(iii)

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

16. Solve the differential equations:
(i)

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\kappa\, N^2(t),$$

(ii)

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} - \frac{2y(x)}{x} = 0,$$

(iii)

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = A x^{2/3}(t); \quad \text{with } \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t) = 0 \text{ at } t = 0$$

(iv)

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} + 2xy(x) = 4x,$$

(v)

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \frac{y(x)}{x} + \tan\left[\frac{y(x)}{x}\right].$$