Cosmology
Lecture 19

Halo mass function

Tirthankar Roy Choudhury
National Centre for Radio Astrophysics
Tata Institute of Fundamental Research
Pune

NCRA • TIFR

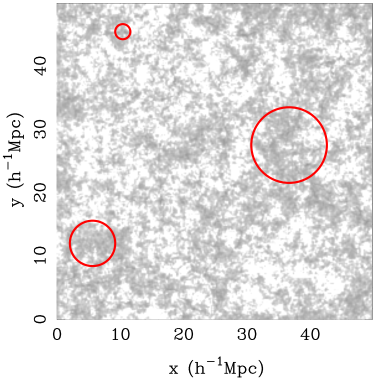
Formation of haloes

- The formation of collapsed objects is crucial for forming galaxies. The first task is to obtain the mass distribution of haloes (i.e., the halo mass function) in the universe for a cosmic density field.
- The full problem is non-linear and cannot be done analytically. However, there is an extremely interesting theoretical model which captures the basics of the formation of haloes.
- We have already seen that, in the spherical approximation, a region collapses and forms a virialized object when the *linear* density contrast within the region exceeds $\delta_c \approx 1.69$.
- Now, suppose we are given the initial density field, i.e., the linear density contrast $\delta(z_{\rm in}, \vec{x})$. We know that it will grow as $D(z)$ in the linear theory.
- Writing $\delta(z, \vec{x}) = D(z)\ \delta(\vec{x})$, where $\delta(\vec{x})$ is the *linearly extrapolated* field at $z = 0$, we understand that a region of comoving radius X will collapse when

$$\delta_X(z, \vec{x}) \geq \delta_c \quad \Longrightarrow \quad \delta_X(\vec{x}) \geq \frac{\delta_c}{D(z)} \equiv \delta_c(z).$$

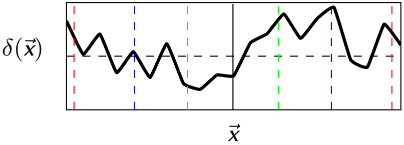
Identification of haloes in the linear field

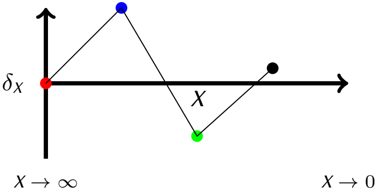
Identify all possible spherical regions which have $\delta_X(\vec{x}) \geq \delta_c(z)$. An extremely cumbersome method!



TRC, Haehnelt & Regan (2009)

Sampling the random field

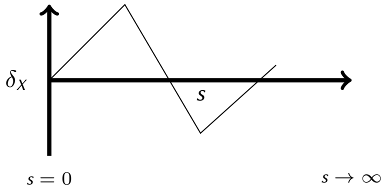




- One can obtain the halo mass function analytically using what is called the *excursion set formalism*.
- Consider the linear density field and let us concentrate on a given point $\vec{x}$.
- Now we smooth the field using a spherical window of radius X. Let us start with a large radius $X \to \infty$ and compute $\delta_X(\vec{x})$. If X is large enough, we expect $\delta_X \to 0$.
- Next we take a smaller radius and compute $\delta_X(\vec{x})$.
- We continue this process with smaller and smaller X.
- The smoothed quantity δ_X seems to be carrying out a "random walk" as a function of smoothing radius X.

Defining the random walk

- The problem can be mapped to a random walk if we make certain modifications to the earlier discussion.
- Let us choose the variable corresponding to random walk steps as $s \equiv \sigma^2(X) = \sigma^2(M)$ (where $M = 4\pi X^3 \bar{\rho}_0/3$) instead of X.
- Note that s is a monotonically decreasing function of X and M. Also $s \to 0$ as $X \to \infty$. Thus all trajectories in the $\delta_X - s$ space start from the origin.



- Each location $\vec{x}$ in the density field $\delta(\vec{x})$ corresponds to a trajectory $\delta_X(s)$, which reflects the value of the density field at that location when smoothed with a filter of radius $X(s)$.

Recap of random walk

- Let $x_1, ..., x_N$ be random variables which can take values ± 1 with probabilities

$$\mathcal{P}(x_i = 1) = \mathcal{P}(x_i = -1) = \frac{1}{2}.$$

  The x_i can be thought of as the distance travelled in a step by a random walker.

- Let $D_N = \sum_i^N x_i$ be the distance after N steps, which itself is a random variable. Then its expectation is

$$\langle D_N \rangle = \sum_{i=1}^N \langle x_i \rangle = \sum_i^N [\mathcal{P}(x_i = 1) \times (+1) + \mathcal{P}(x_i = -1) \times (-1)] = 0.$$

  The symmetry of the probability ensures that the average distance travelled by an ensemble of walkers is zero.

- The variance is given by $\langle D_N^2 \rangle = \sum_{i,j=1}^N \langle x_i x_j \rangle$. If the variables (steps) are uncorrelated, then $\langle x_i x_j \rangle = 0$ when $i \neq j$. In that case

$$\langle D_N^2 \rangle = \sum_{i=1}^N \langle x_i^2 \rangle = N.$$

Filtered density field

- We write the density field smoothed over some length X as $\delta_X(\vec{x}) = \int \mathrm{d}^3 y\, \delta(\vec{y}) W_X(|\vec{y} - \vec{x}|)$.
- In Fourier space, we can write this as $\delta_X(\vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}\, \delta(\vec{k})\ W_X^*(k)$.
- Now, the correlation across different filter scales is given by

$$\langle \delta_{X_1}(\vec{x}) \delta_{X_2}(\vec{x}) \rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \mathrm{e}^{-\mathrm{i}\vec{k}'\cdot\vec{x}} \left\langle \delta(\vec{k}) \delta^*(\vec{k}') \right\rangle W_{X_1}^*(k) W_{X_2}(k') = \int_0^\infty \frac{\mathrm{d}k}{k} \Delta^2(k) W_{X_1}^*(k) W_{X_2}(k).$$

- A special case is

$$\langle \delta_X^2(\vec{x}) \rangle = \int_0^\infty \frac{\mathrm{d}k}{k} \Delta^2(k) W_X^*(k) W_X(k) \equiv \sigma^2(X) \equiv s(X).$$

- Given the above relations, we can make the following correspondence with the random walk problem

Random variable:	$x_i \Longleftrightarrow \delta$
Distance:	$D_N \Longleftrightarrow \delta_X$
Expectation:	$\langle D_N \rangle = 0 \Longleftrightarrow \langle \delta_X \rangle = 0$
Variance:	$\langle D_N^2 \rangle = N \Longleftrightarrow \langle \delta_X^2 \rangle = s$

  Thus $s \equiv \sigma^2$ should play the role of number of steps.
- Note that X, M and s all can be used for measuring the smoothing scale. Hence we will use the notation

$$\delta_X \equiv \delta_s.$$

Correlation between steps

- We still need to show that different steps are uncorrelated for the random walk correspondence to work.
- One can show that this is possible if we use the sharp-k filter

$$W_X(k) = \theta\left(\frac{1}{X} - k\right).$$

- The corresponding filter in real space is

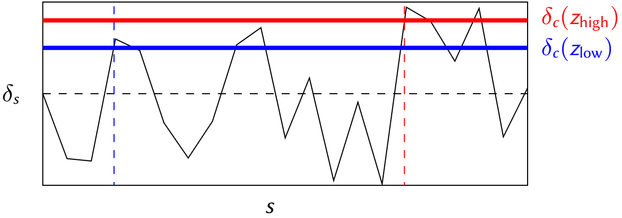
$$W_X(r) = \frac{\sin(r/X) - (r/X)\cos(r/X)}{2\pi^2 r^3}.$$

Barrier crossing

- We have seen that the condition for collapse at redshift z is

$$\delta(\vec{x}) \geq \frac{\delta_c}{D(z)} \equiv \delta_c(z).$$

Thus $\delta_c(z)$ acts like a "barrier" which the random walks must cross for forming haloes.



- The condition for halo formation will be given by the random walks *upcrossing* the barrier for the first time.
- Any subsequent upcrossings would correspond to larger s, i.e., smaller length scales which would simply be structures within the large halo.
- One can see that early redshift implies higher barrier. So statistically the random walks have to travel more at high redshifts to cross the barrier.
- This implies that the haloes which collapse at early times will have larger values of s and hence smaller values of M. Thus small-mass haloes form first. This is known as **hierarchical structure formation**.

Mass function of haloes

- Let a random walk upcross the barrier $\delta_0 = \delta_c / D(z)$ at s_0 for the first time. Then it is expected to form a collapsed object of mass corresponding to the s_0 at a redshift corresponding to $D(z) = \delta_c / \delta_0$.
- Let $f_{\mathrm{FC}}(\delta, s) \mathrm{d}s$ denote the fraction of random walks which *first* upcross the barrier δ at a point between $(s, s + \mathrm{d}s)$.
- If $\delta = \delta_c(z)$, this will be equal to the fraction of points $f(M, z)\mathrm{d}M$ which will collapse to objects with mass $(M, M + \mathrm{d}M)$ at redshift z.
- In the initial field, each of these points has a comoving volume $M(1 + \delta_i)/\bar{\rho}_0 \approx M/\bar{\rho}_0$. So for a large comoving volume V_∞, the total number of points is $V_\infty \bar{\rho}_0 / M$.
- Hence

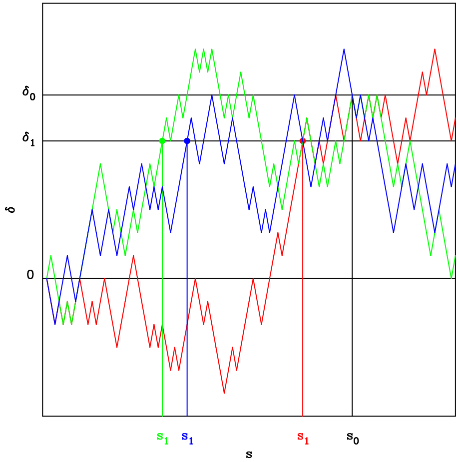
$$f_{\mathrm{FC}}(\delta_c(z), s)\mathrm{d}s = f(M, z)\mathrm{d}M = \frac{1}{\bar{\rho}_0} \frac{M}{V_\infty} N(M, z)\mathrm{d}M = \frac{M}{\bar{\rho}_0} n(M, z)\mathrm{d}M,$$

  or,

$$n(M, z) = \frac{\bar{\rho}_0}{M} f_{\mathrm{FC}}(\delta_c(z), s) \left| \frac{\mathrm{d}s}{\mathrm{d}M} \right| .$$

- This is the number of haloes per unit comoving volume per unit mass range, and is known as the **halo mass function**.

Distribution of first crossing: I



- To obtain $f_{\rm FC}(\delta, s)$, let us first consider the fraction of trajectories $\mathcal{P}(\delta_0, s_0) \mathrm{d}\delta_0$ which have value between $(\delta_0, \delta_0 + \mathrm{d}\delta_0)$ at $s = s_0$. The gaussianity of the linear density field implies

$$\mathcal{P}(\delta_0, s_0) = \frac{1}{\sqrt{2\pi s_0}} \mathrm{e}^{-\delta_0^2/2s_0}.$$

- All of these trajectories must have first upcrossed the point $\delta = \delta_1 < \delta_0$ at some $s = s_1 < s_0$.
- As per our definition, the fraction of points which *first* upcross δ_1 between $(s_1, s_1 + \mathrm{d}s_1)$ is given by $f_{\rm FC}(\delta_1, s_1) \mathrm{d}s_1$.
- Out of these, let a fraction $\mathcal{P}(\delta_0, s_0 | \delta_1, s_1, \mathrm{FC})\ \mathrm{d}\delta_0$ have value between $(\delta_0, \delta_0 + \mathrm{d}\delta_0)$ at s_0.
- Hence, the fraction of points which *first* upcrossed δ_1 between $(s_1, s_1 + \mathrm{d}s_1)$ and have value between $(\delta_0, \delta_0 + \mathrm{d}\delta_0)$ at s_0 is

$$f_{\rm FC}(\delta_1, s_1) \mathrm{d}s_1 \times \mathcal{P}(\delta_0, s_0 | \delta_1, s_1, \mathrm{FC}) \mathrm{d}\delta_0.$$

Distribution of first crossing: II

- The fraction of points which have value between $(\delta_0, \delta_0 + \mathrm{d}\delta_0)$ at s_0 (irrespective of where they first upcrossed δ_1) is obtained by integrating the quantity over s_1, i.e.,

$$\mathcal{P}(\delta_0, s_0)\mathrm{d}\delta_0 = \int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1)\mathcal{P}(\delta_0, s_0|\delta_1, s_1, \mathsf{FC})\mathrm{d}\delta_0.$$

- Now, the trajectories which *first* upcross δ_1 at s_1 can be thought of starting a new random walk from (δ_1, s_1) (instead of $(0, 0)$). This follows from the fact that subsequent steps are uncorrelated with the previous ones.
- Then

$$\begin{aligned}\mathcal{P}(\delta_0, s_0|\delta_1, s_1, \mathsf{FC})\mathrm{d}\delta_0 &= \mathcal{P}(\delta_0 - \delta_1, s_0 - s_1)\mathrm{d}\delta_0 \\ &= \frac{1}{\sqrt{s_0 - s_1}\sqrt{2\pi}}\mathrm{e}^{-(\delta_0 - \delta_1)^2/2(s_0 - s_1)}.\end{aligned}$$

Distribution of first crossing: III

- So, we know $\mathcal{P}(\delta_0, s_0)$ and $\mathcal{P}(\delta_0, s_0|\delta_1, s_1, \mathsf{FC})$ and we want to determine $f_{\mathsf{FC}}(\delta_1, s_1)$.
- To do this, let us integrate over δ_0:

$$\begin{aligned}
\int_{\delta_1}^{\infty} \mathrm{d}\delta_0 \mathcal{P}(\delta_0, s_0) &= \int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1) \int_{\delta_1}^{\infty} \mathrm{d}\delta_0 \mathcal{P}(\delta_0, s_0|\delta_1, s_1, \mathsf{FC}),\\
\frac{1}{\sqrt{2\pi s_0}} \int_{\delta_1}^{\infty} \mathrm{d}\delta_0\ \mathrm{e}^{-\delta_0^2/2s_0} &= \frac{1}{\sqrt{s_0 - s_1}\sqrt{2\pi}} \int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1) \int_{\delta_1}^{\infty} \mathrm{d}\delta_0\ \mathrm{e}^{-(\delta_0-\delta_1)^2/2(s_0-s_1)},\\
\frac{1}{2}\mathrm{erfc}\left(\frac{\delta_1}{\sqrt{2s_0}}\right) &= \frac{1}{\sqrt{s_0 - s_1}\sqrt{2\pi}} \int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1) \int_0^{\infty} \mathrm{d}\bar{\delta}_0\ \mathrm{e}^{-\bar{\delta}_0^2/2(s_0-s_1)}\\
&= \int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1) \times \frac{1}{2}.
\end{aligned}$$

- So,

$$\int_0^{s_0} \mathrm{d}s_1\, f_{\mathsf{FC}}(\delta_1, s_1) = \mathrm{erfc}\left(\frac{\delta_1}{\sqrt{2s_0}}\right).$$

- This can be differentiated to show that

$$f_{\mathsf{FC}}(\delta_1, s_0) = \frac{\delta_1}{\sqrt{2\pi s_0^3}}\mathrm{e}^{-\delta_1^2/2s_0}.$$

The Press-Schechter halo mass function

- Then

$$n(M,z) = \frac{\bar{\rho}_0}{M} f_{\rm FC}(\delta_c(z), s) \left|\frac{\mathrm{d}s}{\mathrm{d}M}\right|$$
$$= \frac{\bar{\rho}_0}{M} \left|\frac{\mathrm{d}s}{\mathrm{d}M}\right| \times \frac{\delta_c(z)}{\sqrt{2\pi s^3}} \mathrm{e}^{-\delta_c^2(z)/2s}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{\bar{\rho}_0}{M} \frac{\delta_c(z)}{s^{3/2}} \left|\frac{\mathrm{d}s}{\mathrm{d}M}\right| \mathrm{e}^{-\delta_c^2(z)/2s}.$$

  This is known as the **Press-Schechter mass function**.
- One can improve the model by incorporating collapse of ellipsoids. This leads to an improvement to the match with simulations, and the mass function is known as the **Sheth-Tormen mass function**.

Comparison with simulations