Cosmology
Lecture 17

Statistical description of the perturbations

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Statistics of density perturbations

- In the previous lectures, we have studied the temporal growth of density perturbations – using both linear and non-linear theory. Now, we shall study the spatial distribution of the perturbations at different scales.
- In general, to specify the cosmological density field at any epoch t, one needs to know the value of $\rho(\vec{x})$ at all spatial points $\vec{x}$. This is impossible since it requires knowledge of an infinite number of field values.
- Although we observe structures in the large-scale distribution of galaxies, there is no obvious pattern in the density field.
- The cosmic density field is thus believed to be a random field generated by some stochastic processes, hence one should study only the statistical properties of the cosmic density field. The density field we observe is simply one specific realization of the underlying random field.
- The situation is similar to that in statistical mechanics where we are not interested in the positions and momenta of individual particles, rather we study the statistical properties of the system using some distribution function.

Ensembles of the cosmic density field

- Just like in statistical mechanics, it is convenient to define the statistical properties in terms of the ensemble averages.
- Consider a ensemble of universe denoted by $\alpha = 1, 2, \ldots, N_{\rm en}$, each with a density field $\rho_{(\alpha)}(\vec{x})$.
- Its *ensemble average* at a point $\vec{x}_i$ is denoted as

$$\langle \rho(\vec{x}_i) \rangle = N_{\rm en}^{-1} \sum_{\alpha=1}^{N_{\rm en}} \rho_{(\alpha)}(\vec{x}_i).$$

- If the universe is *statistically homogeneous*, then the ensemble average should be independent of position:

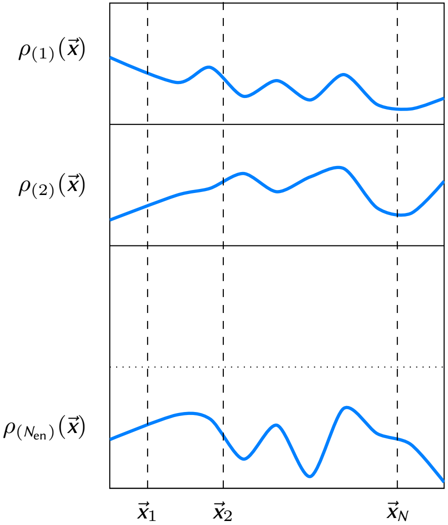
$$\langle \rho(\vec{x}_i) \rangle = \langle \rho \rangle.$$

- Now, in practice one cannot measure the ensemble averages like $\langle \rho \rangle$. However, one can evaluate volume averages like

$$\rho_V(\vec{x}) = \frac{1}{V} \int_V \mathrm{d}^3 x' \; \rho(\vec{x} + \vec{x}'),$$

where V is volume centered at $\vec{x}$.

- How is $\rho_V(\vec{x})$ related to $\langle \rho \rangle$?



Ergodicity

- Clearly the ensemble average of the volume average is given by

$$\langle \rho_V(\vec{x}) \rangle = \frac{1}{V} \int_V \mathrm{d}^3 x' \, \left\langle \rho(\vec{x} + \vec{x'}) \right\rangle = \langle \rho \rangle \frac{1}{V} \int_V \mathrm{d}^3 x' = \langle \rho \rangle.$$

  Hence the volume average is an *unbiased estimator* of $\langle \rho \rangle$.
- However, the volume average will be useful only if the variance of the quantity among various ensembles is small. In other words, we should try to determine the conditions under which the variance $\left\langle (\rho_V(\vec{x}) - \langle \rho \rangle)^2 \right\rangle$ is small.
- It can be shown that the variance $\left\langle (\rho_V(\vec{x}) - \langle \rho \rangle)^2 \right\rangle$ goes to zero as $V \to \infty$ (length larger than the correlation scale).
- Thus the quantity ρ_V can be a good estimator of $\langle \rho \rangle$ provided V is large, i.e.,

$$\lim_{V \to \infty} \rho_V(\vec{x}) = \lim_{V \to \infty} \frac{1}{V} \int_V \mathrm{d}^3 x' \, \rho(\vec{x} + \vec{x'}) \quad \Longleftrightarrow \quad \langle \rho \rangle.$$

- This corresponds to the equality between ensemble and volume averages, similar to the ergodicity used in statistical mechanics.
- We will write

$$\lim_{V \to \infty} \rho_V = \lim_{V \to \infty} \frac{1}{V} \int_V \mathrm{d}^3 x \, \rho(\vec{x}) \equiv \bar{\rho}.$$

- One can also show that as $V \to \infty$, all other relevant statistical quantities also satisfy the ergodicity.
- We can also define the density contrast as earlier $\delta(\vec{x}) \equiv \dfrac{\rho(\vec{x})}{\bar{\rho}} - 1$. Note that $\langle \delta \rangle = \bar{\delta} = 0$.

Probabilistic interpretation

- Let us consider a set of N point objects (say, galaxies) of mass m. Then the density is given by

$$\rho(\vec{x}) = m \sum_{i=1}^{N} \delta_D(\vec{x} - \vec{x}_i).$$

- Clearly, the volume average is as expected given by

$$\bar{\rho} = \frac{1}{V} \int_V \mathrm{d}^3x \, \rho(\vec{x}) = \frac{mN}{V} = mn.$$

- The probability of finding a mass point (galaxy) in a randomly chosen volume ΔV at $\vec{x}$ is given by

$$\mathcal{P}_1 = n\Delta V = \frac{\langle \rho(\vec{x}) \rangle \Delta V}{m}.$$

- The joint probability of finding a galaxy in ΔV_1 and a galaxy in ΔV_2 is

$$\mathcal{P}_{12} = \frac{\langle \rho(\vec{x}_1)\rho(\vec{x}_2)\rangle \Delta V_1 \Delta V_2}{m^2} = \frac{n^2 \langle \rho(\vec{x}_1)\rho(\vec{x}_2)\rangle}{\bar{\rho}^2} \Delta V_1 \Delta V_2 = n^2 \left[1 + \xi(\vec{x}_1, \vec{x}_2)\right] \Delta V_1 \Delta V_2,$$

  where

$$\xi(\vec{x}_1, \vec{x}_2) \equiv \frac{\langle \rho(\vec{x}_1)\rho(\vec{x}_2)\rangle}{\bar{\rho}^2} - 1$$

  measures the excess probability, over random, of finding two galaxies at the volume elements ΔV_1 and ΔV_2.
- The quantity ξ is called the **two-point correlation function**.

Two-point correlation function

- In terms of the density contrast, we have

$$\xi(\vec{x}_1, \vec{x}_2) = \left\langle \frac{\rho(\vec{x}_1)}{\bar{\rho}} \frac{\rho(\vec{x}_2)}{\bar{\rho}} \right\rangle - 1 = \langle [1 + \delta(\vec{x}_1)] \, [1 + \delta(\vec{x}_2)] \rangle - 1 = \langle \delta(\vec{x}_1) \delta(\vec{x}_2) \rangle.$$

- The statistical homogeneity would imply that the correlation function should depend only on the separation between the two points, i.e.,

$$\xi(\vec{x}_1, \vec{x}_2) = \xi(\vec{x}_1 - \vec{x}_2).$$

- Similarly, statistical isotropy would imply that

$$\xi(\vec{x}_1 - \vec{x}_2) = \xi(|\vec{x}_1 - \vec{x}_2|),$$

i.e., the correlation does not depend on the direction of the separation between the two points.

Fourier transform

- The Fourier transform of the density contrast is

$$\delta(\vec{k}) = \int \mathrm{d}^3x\, \delta(\vec{x})\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}.$$

  Note that $\delta^*(\vec{k}) = \delta(-\vec{k})$ because $\delta(\vec{x})$ is real.
- Then

$$\begin{aligned}\left\langle \delta(\vec{k})\, \delta^*(\vec{k}') \right\rangle &= \left\langle \int \mathrm{d}^3x\, \delta(\vec{x})\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} \int \mathrm{d}^3x'\, \delta(\vec{x}')\mathrm{e}^{\mathrm{i}\vec{k}'\cdot\vec{x}'} \right\rangle \\ &= \int \mathrm{d}^3x \int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')} \langle \delta(\vec{x})\delta(\vec{x}')\rangle \\ &= \int \mathrm{d}^3x \int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')}\, \xi(\vec{x}, \vec{x}').\end{aligned}$$

- From statistical homogeneity, we get $\xi(\vec{x}, \vec{x}') = \xi(\vec{x} - \vec{x}')$

$$\left\langle \delta(\vec{k})\, \delta^*(\vec{k}') \right\rangle = \int \mathrm{d}^3x \int \mathrm{d}^3x' \mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')} \xi(\vec{x} - \vec{x}') = (2\pi)^3\, \delta_D(\vec{k} - \vec{k}') \int \mathrm{d}^3y' \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{y}'} \xi(\vec{y}').$$

Power spectrum

- Define the **power spectrum** as the Fourier transform of the correlation function

$$P(\vec{k}) = \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\, \xi(\vec{x}),$$

so that

$$\left\langle \delta(\vec{k})\, \delta^*(\vec{k'}) \right\rangle = (2\pi)^3 \delta_D(\vec{k} - \vec{k'}) P(\vec{k}).$$

Thus the different $\vec{k}$-modes are *statistically uncorrelated* for a statistically homogeneous universe.

- Now from statistical isotropy, we get $\xi(\vec{x}) = \xi(|\vec{x}|)$. Hence

$$P(\vec{k}) = \int \mathrm{d}^3x\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\, \xi(x) = 2\pi \int_0^\infty \mathrm{d}x\, x^2\, \xi(x) \int_{-1}^1 \mathrm{d}\mu\, \mathrm{e}^{-\mathrm{i}kx\mu} = \int_0^\infty \mathrm{d}x\, \left[4\pi x^2 \xi(x)\right] \left(\frac{\sin kx}{kx}\right),$$

showing that $P(\vec{k}) = P(k)$.

- The correlation function can be written as a inverse Fourier transform

$$\xi(x) \equiv \langle \delta(\vec{y})\, \delta(\vec{y} - \vec{x}) \rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} P(k)\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} = \int_0^\infty \frac{\mathrm{d}k}{k} \left[\frac{k^3 P(k)}{2\pi^2}\right] \left(\frac{\sin kx}{kx}\right).$$

- The quantity

$$\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}$$

is called the **dimensionless power spectrum**. It measures the power contained within logarithmic intervals in k.

Smoothed density contrast

- We can smooth the density contrast over some length scale X using window functions

$$\delta_X(\vec{x}) = \int \mathrm{d}^3 y\, \delta(\vec{y})\ W_X(\vec{y} - \vec{x}) = \int \mathrm{d}^3 y\, \delta(\vec{x} + \vec{y})\ W_X(\vec{y}).$$

- This implies $\delta_X(\vec{k}) = \delta(\vec{k})\ W_X(-\vec{k})$.
- One example of the window function is the spherical top-hat filter

$$W_X(y) = \left(\frac{4\pi X^3}{3}\right)^{-1} \theta(X - |\vec{y}|).$$

  In this case

$$\delta_X(\vec{x}) = \left(\frac{4\pi X^3}{3}\right)^{-1} \int \mathrm{d}^3 y\, \delta(\vec{x} + \vec{y})\theta(X - |\vec{y}|),$$

  where the theta function ensures that the integral is over a sphere of radius X.
- The Fourier transform of the spherical top hat filter is

$$W_X(k) = \int \mathrm{d}^3 x\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} W_X(x) = \frac{3(\sin kX - kX \cos kX)}{k^3 X^3}.$$

- Other examples of window functions are Gaussian and sharp-k filters.

Fluctuations averaged over a scale

- The density fluctuations averaged over a window function is

$$\sigma^2(X) \equiv \langle \delta_X^2 \rangle = \left\langle \int \mathrm{d}^3 y\, \delta(\vec{x}+\vec{y}) W_X(\vec{y}) \int \mathrm{d}^3 y'\, \delta(\vec{x}+\vec{y'}) W_X(\vec{y'}) \right\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P(k)\, |W_X(\vec{k})|^2.$$

- For example, the density fluctuations averaged over a sphere of radius X is given by

$$\langle \delta_X^2 \rangle = \sigma^2(X) = \int_0^\infty \frac{\mathrm{d}k}{k} \left[\frac{k^3 P(k)}{2\pi^2}\right] \left[\frac{3(\sin kX - kX \cos kX)}{k^3 X^3}\right]^2.$$

  Note that $\sigma^2 \equiv \sigma^2(0) = \xi(0) = \langle \delta^2 \rangle$.
- The function $\sigma(X)$ is usually a monotonic function of X at scales of interest, it decreases with increasing X.
- The quantity $\sigma(X)$ can be used to fix the normalization of the power spectrum. If we write $P(k) = A_s k^n T^2(k)$, then the fluctuations within a sphere of radius X will be

$$\sigma^2(X) = A_s \int_0^\infty \frac{\mathrm{d}k}{k} \left[\frac{k^{3+n}}{2\pi^2}\right] T^2(k) \left[\frac{3(\sin kX - kX \cos kX)}{k^3 X^3}\right]^2.$$

- It is possible, from measurements of cluster abundances, to constrain the value of $\sigma_8 \equiv \sigma(X = 8h^{-1}\ \mathrm{Mpc})$. One can then find the value of A_s and thus the power spectrum is fully determined.
- An alternate way of normalizing the power spectrum is to use the CMB fluctuations measured by COBE (often called COBE-normalization).

Probability distribution of a random field

- It is quite difficult to specify a general random field, because it involves the determination of an infinite number of quantities.
- For simplicity let us divide the volume into n infinitesimal cells which are centred at $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. In the continuum limit we take $n \to \infty$ and the cell volumes go to zero.
- The random perturbation field $\delta(\vec{x})$ is then characterized by the set of n numbers $\delta_1, \delta_2, \ldots, \delta_n$, where $\delta_i \equiv \delta(\vec{x}_i)$.
- To specify the field in a statistical sense, we need to specify the probability distribution function

  $$\mathcal{P}(\delta_1, \delta_2, \ldots, \delta_n) \, \mathrm{d}\delta_1 \, \mathrm{d}\delta_2 \ldots \mathrm{d}\delta_n,$$

  which gives the probability that the field δ has values in the range δ_i to $\delta_i + \mathrm{d}\delta_i$ at positions $\vec{x}_i$ $(i = 1, 2, \ldots, n)$.
- In general, we need to specify infinite number of moments to determine $\mathcal{P}$.
- Fortunately, the initial linear density perturbation field in the Universe is found to be well approximated by a homogeneous and isotropic Gaussian random field which is completely determined, in a statistical sense, by its power spectrum or its two-point correlation function.

Gaussian random field

- For a set of *linear* density contrasts $\delta(\vec{x}_i) = \delta_i$, the joint probability distribution is given by
$$\mathcal{P}(\delta_1, ..., \delta_n) = \frac{1}{(2\pi)^{n/2}|M|^{1/2}} \exp\left[-\frac{1}{2}\delta^T \cdot M^{-1} \cdot \delta\right],$$
where
$$M_{ij} = \langle \delta_i \delta_j \rangle.$$
Note that $\langle \delta_i \rangle = 0$.
- Note that the δ's at two points are correlated through the off-diagonal terms of M_{ij}.
- The one point distribution is given by
$$\mathcal{P}(\delta) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\delta^2/2\sigma^2},$$
i.e., it is completely determined by the variance $\langle \delta^2 \rangle = \sigma^2$.
- Note that the smoothed density field δ_X is just a sum of many Gaussian random variables δ_i, thus it too is a Gaussian random field. The variance is $\sigma^2(X)$, and hence the one point distribution function is
$$\mathcal{P}(\delta_X) = \frac{1}{\sigma(X)\sqrt{2\pi}} \mathrm{e}^{-\delta_X^2/2\sigma^2(X)}.$$
- The Fourier modes $\delta(\vec{k})$ too are sum of Gaussian random variables δ_i and thus form a Gaussian random field. However, they are uncorrelated, i.e., $\left\langle \delta(\vec{k})\delta^*(\vec{k}') \right\rangle = 0$ for $\vec{k} \neq \vec{k}'$. Hence only the diagonal terms in the covariance matrix survive and they are essentially given by $P(k)$.

Linearly extrapolated fields

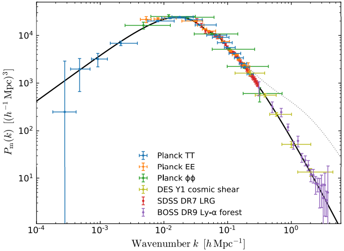
- In the linear regime, we know that $\delta \propto D(z)$ (both in the real and Fourier space).
- Suppose we extrapolate this evolution up to $z = 0$ (when the linear approximation is not valid). Then we can write

$$\delta(z, \vec{k}) = D(z)\, \delta(z = 0, \vec{k}).$$

  This is known as the *linearly extrapolated density field.*
- The *linearly extrapolated power spectrum* is

$$P(z, k) = D^2(z)\; P(z = 0, k) = D^2(z)\; P(k),$$

  where $P(k) = P(z = 0, k)$ is the power spectrum *linearly extrapolated* to $z = 0$.



Courtesy ESA and Planck Collaboration

- A similar relation can be written for the fluctuations in real space, e.g., $\sigma^2(z, X) = D^2(z)\, \sigma^2(X)$.

The CMB fluctuations

- The same formalism can be applied to fields defined on the surface of the sky, e.g., the CMB. In that case, rather than using the three-dimensional Fourier transform, one can use the spherical harmonic transforms.
- The temperature fluctuations $\Theta(\theta, \phi) \equiv \delta T(\theta, \phi)/T_0$, defined on the surface, can be decomposed as

$$a_{\ell m} = \int \mathrm{d}\Omega \; Y^*_{\ell m}(\theta, \phi) \; \Theta(\theta, \phi),$$

  with the inverse transform being

$$\Theta(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi).$$

- The power spectrum of temperature fluctuations is

$$C_\ell = \left\langle |a_{\ell m}|^2 \right\rangle .$$

- The correlation function is

$$C(\vartheta) \equiv \langle \Theta(\theta_1, \phi_1) \; \Theta(\theta_2, \phi_2) \rangle = \frac{1}{4\pi} \sum_\ell (2\ell + 1) \; C_\ell \; P_\ell(\cos\vartheta).$$