Cosmology
Lecture 16
Spherical collapse

Tirthankar Roy Choudhury
National Centre for Radio Astrophysics
Tata Institute of Fundamental Research
Pune

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Spherical evolution

- For dark matter, a simple but effective approximation scheme, based on spherical symmetry, can be used for studying the dynamics of a overdense spherical region. As the universe expands, the overdense region will expand more slowly compared to the background, will reach a maximum radius, contract and virialize to form a bound non-linear system.
- Let us assume that collapsing shells within the overdense region do not cross. In that case, the mass M enclosed by the shell will not change.
- The equation of motion for a particle on the shell of proper radius R in an expanding background will be

$$\ddot{R} = -\frac{GM}{R^2} - \frac{4\pi G}{3}(\bar{\rho}_{\rm rest} + 3\bar{P}_{\rm rest})R,$$

  where we include other components of energy density $\bar{\rho}_{\rm rest} + 3\bar{P}_{\rm rest}$, if present, through their gravitational acceleration.
- For a universe with dark matter and a cosmological constant, we have $\bar{\rho}_{\rm rest} = -\bar{P}_{\rm rest} = \Omega_\Lambda(3H_0^2/8\pi G)$, then

$$\ddot{R} = -\frac{GM}{R^2} + H_0^2\Omega_\Lambda R.$$

- Clearly, the effect of Λ becomes important only for large R, corresponding to densities $\sim \rho_{c,0}$. For the most part, we will be studying high-density collapsed structures, so we will ignore the effect of Λ.

Initial conditions

- The second order equation can be solved for a given background cosmology and a mass scale M, provided the initial conditions are known.
- The initial velocity is obtained by assuming that at very early epochs ($a = a_i \sim 10^{-3}$) the shell was expanding very nearly with the background universe – hence its initial velocity can be approximated by the Hubble expansion velocity at that epoch $\dot{R}_i \approx H(a_i) R_i$.
- The initial value of R is still arbitrary – however, it should be chosen such that the initial density contrast is much less than unity (so that the linear approximation is valid).
- Recall that the density contrast is given by

$$\delta(t) = \frac{2GM}{\Omega_{m,0} H_0^2} \frac{a^3(t)}{R^3(t)} - 1.$$

  We need $0 < \delta(t_i) \ll 1$.
- Keep in mind that the background solution for a matter-dominated universe is simply

$$a(t) = \left(\frac{3H_0 t}{2} \right)^{2/3}, \quad \frac{\dot{a}}{a} = \frac{2}{3t}.$$

  Then the density contrast evolves as

$$\delta(t) = \frac{9GM}{2} \frac{t^2}{R^3} - 1.$$

The first integral

- The equation of motion has a first integral
$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = E,$$
where E is a constant. The solution will be bound only when $E < 0$.
- This can be compared with the Friedmann equations which govern the evolution of the background universe:
$$\frac{\ddot{a}}{a} = -\frac{H_0^2}{2}\frac{\Omega_{m,0}}{a^3}, \qquad \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = H_0^2\frac{\Omega_{m,0}}{a^3},$$
which can also be written as
$$\ddot{a} = -\frac{H_0^2\Omega_{m,0}}{2a^2}, \qquad \frac{1}{2}\dot{a}^2 - \frac{H_0^2\Omega_{m,0}}{2a} = -\frac{k}{2}.$$
Thus, the spherical collapse problem is exactly like cosmology in a spatially curved (closed) universe.
- We already know that the solution will be of the form of a cycloid.
- The energy E, which is conserved during the motion, can be related to the initial conditions at a very early epoch a_i. We assume that $\delta_i \ll 1$ and $\dot{R}_i \approx H_i R_i$ (the shell is expanding with the Hubble flow). Then
$$\begin{aligned} E &= \frac{1}{2}\dot{R}_i^2 - \frac{GM}{R_i} \approx \frac{1}{2}H_i^2 R_i^2 - \frac{GM}{R_i} \\ &= -\frac{GM}{R_i}\left(1 - \frac{H_i^2 R_i^3}{2GM}\right) = -\frac{GM}{R_i}\left(1 - \frac{H_0^2\Omega_{m,0} R_i^3}{2GMa_i^3}\right) = -\frac{GM}{R_i}\left(1 - \frac{1}{1+\delta_i}\right), \end{aligned}$$
$$E \approx -\frac{GM}{R_i}\delta_i.$$

Turnaround

- Let us first calculate the maximum (turnaround) radius which corresponds to $\dot{R} = 0$

$$R_{\rm ta} = -\frac{GM}{E}.$$

Clearly, the turnaround radius is physical only when $E < 0$. If $E > 0$, the condition $\dot{R} = 0$ will never be satisfied.

- We already have $E \approx -GM/R_i \times \delta_i$. Thus, $E < 0$ when $\delta_i > 0$, i.e., regions which are initially overdense will eventually turnaround and collapse.
- So the turnaround radius is related to the initial radius by simply:

$$R_{\rm ta} = \frac{R_i}{\delta_i}.$$

Solutions to the evolution equation

- The solutions of the equations of motion are simply given by the parametric form

$$R = A(1 - \cos\theta), \quad t = B(\theta - \sin\theta), \quad A = \frac{GM}{2|E|}, \; B = \frac{GM}{(2|E|)^{3/2}}, \; A^3 = GMB^2.$$

- The evolution of the density contrast is then

$$\delta = \frac{9GM}{2}\frac{B^2(\theta - \sin\theta)^2}{A^3(1 - \cos\theta)^3} - 1 = \frac{9(\theta - \sin\theta)^2}{2(1 - \cos\theta)^3} - 1,$$

which shows that $\delta(\theta)$ is independent of M.

- Now let us check whether everything makes sense for early times $t \to 0$. This corresponds to $\theta \to 0$.
- It is easy to show that $R \propto t^{2/3}$ at early times, showing that the radius of the region is expanding with the Hubble flow.
- It can also be shown that at $t \to 0$, the density contrast evolves as

$$\lim_{t \to 0} \delta(t) \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} \equiv \delta_L(t).$$

This is also expected since $a \propto t^{2/3}$ and $\delta \propto a$ in linear theory. We have denoted the linear evolution of the density contrast as $\delta_L(t)$.

Turnaround overdensity

- For turnaround, we have $\theta = \pi$ where R is maximum, so that

$$R_{\text{ta}} = 2A = \frac{GM}{|E|}, \quad t_{\text{ta}} = \pi B = \frac{\pi GM}{(2|E|)^{3/2}},$$

  which is same as what we got earlier.

- Also,

$$\delta_{\text{ta}} = \frac{9\pi^2}{16} - 1 \approx 4.55,$$

  and

$$\delta_{L,\text{ta}} = \frac{3}{20}(6\pi)^{2/3} \approx 1.06.$$

- Thus, the turnaround occurs when the linear density contrast is 1.06, by that time the actual density contrast is as high as 4.55.

Collapse of the sphere

- After turnaround, the shell collapses with its density increasing rapidly. The solution gives that the region will simply collapse to a point ($R \to 0$ corresponding to $\theta \to 2\pi$) at an epoch $t_{\rm vir}$.
- However, one should realize that the simple idealized model breaks down at such high densities because small non-radial motions get amplified and shells start crossing each other.
- The system relaxes violently and forms a virialized bound structure. The formalism mentioned here is not adequate to describe evolution during the period when the system relaxes ($t \sim t_{\rm vir}$).
- After virialization, the system attains a steady state. In that case we can use the **virial theorem** to relate the kinetic and potential energy.
- We can write $K = -U/2$ and hence $E = K + U = U/2$.

Properties of the virialized object

- One can determine the properties of the system when it has already virialized assuming that the system virializes at $\theta = 2\pi$

$$t_{\rm vir} = 2\pi B = \frac{\pi G M}{\sqrt{2}|E|^{3/2}} \implies \delta_{L,\rm vir} = \frac{3}{20}(12\pi)^{2/3} \approx 1.686.$$

- The value of the linear density contrast when the sphere virializes is called the **critical density of collapse** and is written as δ_c.
- For a matter-dominated universe, we have computed its value. For a ΛCDM model, the value differs from 1.686 but can be calculated numerically.
- To compute the radius, we use

$$E = \frac{U}{2} = -\frac{GM}{2R_{\rm vir}} \implies R_{\rm vir} = A = \frac{GM}{2|E|} = \frac{R_{\rm ta}}{2}.$$

- Then the actual overdensity of the system is

$$\Delta_{\rm vir} \equiv 1 + \delta_{\rm vir} = \frac{9GM}{2}\frac{t_{\rm vir}^2}{R_{\rm vir}^3} = \frac{9GM}{2}\frac{4\pi^2 B^2}{A^3} = 18\pi^2 \approx 177.65.$$

- Interestingly, $\Delta_{\rm vir}$ is independent of the properties of the system.

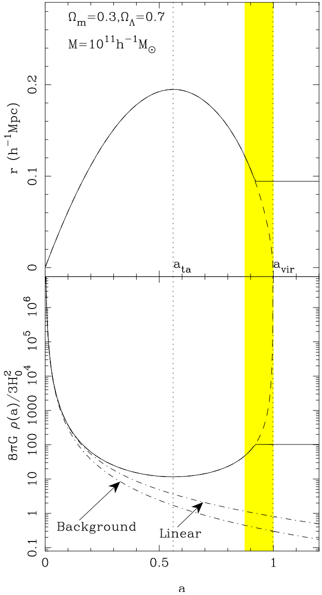
Relaxation time-scale

- One may think that the epoch virialization should not be that corresponding to the collapse, but the one when the radius is $R_{\rm ta}/2$ qhich implies $\theta = 3\pi/2$. This corresponds to a time $B(3\pi/2 + 1)$ which is $\sim B(\pi/2 - 1)$ smaller than the time of collapse.
- However, the system takes some time to relax after $\theta = 3\pi/2$, the time scale being given by the free fall time scale

$$t_{\rm ff} \sim \frac{1}{\sqrt{G\rho}} \sim \sqrt{\frac{R_{\rm vir}^3}{GM}} = \sqrt{\frac{A^3}{GM}} = B$$

  which is similar to the time difference. Thus it is assumed that the collapse happens close to the collapse time $\theta = 2\pi$.

Plot of the solution



Scalings and other relations

- Since the density at collapse is a constant, we can obtain a relation between the mass and the virial radius

$$\frac{M}{10^{11} h^{-1} M_\odot} = \left(\frac{R_{\rm vir}}{117 h^{-1} {\rm kpc}}\right)^3 \frac{\Omega_{m,0}}{0.3} \frac{\Delta_{\rm vir}}{18\pi^2} (1 + z_{\rm vir})^3 .$$

- In practice, one is often interested in the circular velocity v_c of the collapsed halo. Since $K = -U/2$ at virialization, we get

$$v_c^2 = \frac{GM}{R_{\rm vir}},$$

  where we have assumed that the virialized halo has a singular isothermal density profile $\rho(r) \propto r^{-2}$.
- One can see that $M \propto R_{\rm vir}^3$ and $v_c^2 \propto M/R_{\rm vir} \propto M^{2/3}$.
- The above relations can be used to relate the circular velocity to the mass of the halo

$$\frac{M}{10^{11}\ h^{-1} M_\odot} = \left(\frac{v_c}{60.49\ {\rm km\ s}^{-1}}\right)^3 \left(\frac{0.3}{\Omega_{m,0}}\right)^{1/2} \left(\frac{18\pi^2}{\Delta_{\rm vir}}\right)^{1/2} (1 + z_{\rm vir})^{-3/2}$$

- For other cosmologies, the relations are modified slightly. The problem has to be solved numerically – the qualitative features, however, remain the same.

Baryons and shock heating

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- Although the evolution of baryons were not included in the solutions, one can infer their behaviour in a qualitative manner.
- The gas will essentially follow the gravitational attraction of the dark matter and will settle in the collapsed halo.
- The gas, when attracted into the dark matter potential well, acquires kinetic energy and thus gets heated up. The typical temperature of the gas (assuming it to be hydrogen) would be

$$T_{\rm vir} \sim \frac{G M m_p}{k_B R_{\rm vir}}.$$

  Clearly $T_{\rm vir} \propto M^{2/3}$.
- This temperature is $\sim 4.5 \times 10^5$ K for $10^{11} h^{-1} M_\odot$ dark matter haloes, and is smaller for lighter haloes.
- The gas, thus shock heated, cannot condense unless some cooling mechanism takes over. This forms the basis of galaxy formation studies.