Cosmology
Lecture 14

Newtonian perturbation theory

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Fluid equations

- The evolution of cosmological perturbations is quite a complicated exercise in linearized general relativity. However, the most interesting scales, where formation of structures takes place in the post-recombination era, are much smaller than the Hubble length $c/H(z)$.
- For such scales, relativistic effects can be ignored and most of the essential physics can be extracted from a Newtonian approach. We can treat the dark matter and baryons as fluids, their properties being governed by the non-relativistic equations of fluid dynamics.
- The fundamental equations governing fluid motion are

$$\dot{\rho}(t, \vec{r}) + \vec{\nabla}_r \cdot [\rho(t, \vec{r})\ \vec{U}(t, \vec{r})] = 0 \qquad \text{(Continuity equation)}$$

$$\dot{\vec{U}}(t, \vec{r}) + [\vec{U}(t, \vec{r}) \cdot \vec{\nabla}_r]\vec{U}(t, \vec{r}) = -\vec{\nabla}_r \Phi(t, \vec{r}) - \frac{\vec{\nabla}_r P(t, \vec{r})}{\rho(t, \vec{r})} \qquad \text{(Euler equation)}$$

$$\vec{\nabla}_r^2 \Phi(t, \vec{r}) = 4\pi G \rho(t, \vec{r}) \qquad \text{(Poisson equation)}$$

where

- the overdot represents partial derivative $\partial/\partial t$,
- $\vec{\nabla}_r$ is the spatial gradient operator with respect to the proper coordinates $\vec{r}$,
- the fluid density and pressure are denoted by $\rho(t, \vec{r})$ and $P(t, \vec{r})$, respectively,
- the proper velocity of the fluid is $\vec{U}(t, \vec{r}) \equiv \mathrm{d}\vec{r}/\mathrm{d}t$,
- the quantity $\Phi(t, \vec{r})$ is the gravitational potential.

Comoving coordinates

- The equations can be rewritten in terms of the comoving coordinate $\vec{x}$ defined by

$$\vec{r} = a(t)\vec{x}.$$

- The comoving coordinates label observers who follow the Hubble expansion in an unperturbed universe (i.e., $\vec{x}$ would not change for these observers if the universe in unperturbed). Hence, the large-scale expansion is divided out in the comoving coordinates, the only way they change is because of irregularities.
- We then have

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla}_x,$$

and

$$\vec{U} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{a}\vec{x} + a\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \frac{\dot{a}}{a}\vec{r} + a\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}.$$

- The quantity $\vec{v} \equiv a\,\mathrm{d}\vec{x}/\mathrm{d}t$ is the **peculiar velocity**. The first part $(\dot{a}/a)\,\vec{r}$ is the "Hubble velocity".
- The physical density can be written in terms of the comoving density ρ_0 as

$$\rho = \frac{\rho_0}{a^3}.$$

Perturbed quantities

- We can also divide out the smooth component of other quantities and write the equations in terms of the perturbed quantities, namely,

$$
\begin{aligned}
\text{Density contrast} \qquad & \delta(t, \vec{x}) \equiv \frac{\rho(t, \vec{x})}{\bar{\rho}(t)} - 1 \\
\text{Peculiar velocity field} \qquad & \vec{v}(t, \vec{x}) \equiv a(t) \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{U}(t, \vec{x}) - \frac{\dot{a}}{a} \vec{r} \\
\text{Perturbed pressure} \qquad & p(t, \vec{x}) = P(t, \vec{x}) - \bar{P}(t) \\
\text{Perturbed gravitational field} \qquad & \phi(t, \vec{x}) = \Phi(t, \vec{x}) - \bar{\Phi}(t, \vec{x}).
\end{aligned}
$$

- The symbols with bars denote the average values of the corresponding quantities, which are independent of the spatial coordinates except $\bar{\Phi}$, which satisfies the equation for the smooth universe

$$
\vec{\nabla}_r^2 \bar{\Phi} = 4\pi G \bar{\rho}.
$$

Fluid equations in terms of the perturbed quantities

- While writing the equations in terms of the perturbed quantities and comoving coordinates, the crucial point to note is that the time derivative $\partial/\partial t$ has to be modified while changing the coordinates from $\vec{r} \to \vec{x}$, i.e., $\partial/\partial t \to \partial/\partial t - (\dot{a}/a)\vec{x} \cdot \vec{\nabla}_x$ whenever we write the equations in terms of $\vec{x}$.
- In terms of these perturbed quantities, the perturbed fluid equations (i.e., after subtracting out the zeroth order unperturbed part) become

$$\dot{\delta} + \frac{1}{a}\,\vec{\nabla} \cdot [(1+\delta)\,\vec{v}] = 0,$$
$$\dot{\vec{v}} + \frac{\dot{a}}{a}\,\vec{v} + \frac{1}{a}\,(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{a}\,\vec{\nabla}\phi - \frac{\vec{\nabla} p}{a\bar{\rho}(1+\delta)},$$
$$\vec{\nabla}^2 \phi = 4\pi G \bar{\rho} a^2 \delta,$$

where we are using the convention

$$\vec{\nabla} \equiv \vec{\nabla}_x.$$

Dark matter and baryons

- To study in full detail, one has to solve the fluid equations for dark matter and baryons separately.
- In order to do this, it is assumed that $p_{\rm DM} = 0$ for the collisionless dark matter.
- However, since the baryons collide among themselves and interact with radiation, one cannot neglect the corresponding pressure term $p_b \propto \rho_b k_B T$.
- Thus the equations for dark matter and baryons become

$$\dot{\delta}_{\rm DM} + \frac{1}{a} \vec{\nabla} \cdot [(1 + \delta_{\rm DM}) \, \vec{v}_{\rm DM}] = 0,$$

$$\dot{\vec{v}}_{\rm DM} + \frac{\dot{a}}{a} \vec{v}_{\rm DM} + \frac{1}{a} (\vec{v}_{\rm DM} \cdot \vec{\nabla}) \vec{v}_{\rm DM} = -\frac{1}{a} \vec{\nabla} \phi,$$

$$\dot{\delta}_b + \frac{1}{a} \vec{\nabla} \cdot [(1 + \delta_b) \, \vec{v}_b] = 0,$$

$$\dot{\vec{v}}_b + \frac{\dot{a}}{a} \vec{v}_b + \frac{1}{a} (\vec{v}_b \cdot \vec{\nabla}) \vec{v}_b = -\frac{1}{a} \vec{\nabla} \phi - \frac{\vec{\nabla} p_b}{a \bar{\rho}_b (1 + \delta_b)},$$

$$\vec{\nabla}^2 \phi = 4\pi G a^2 (\bar{\rho}_{\rm DM} \delta_{\rm DM} + \bar{\rho}_b \delta_b) = \frac{3}{2} \frac{H_0^2}{a} \left(\Omega_{{\rm DM},0} \delta_{\rm DM} + \Omega_{b,0} \delta_b\right),$$

where, in the last equation, we have used

$$\bar{\rho}_{\rm DM}(t) = \frac{\bar{\rho}_{{\rm DM},0}}{a^3(t)} = \frac{\Omega_{{\rm DM},0} \rho_{c,0}}{a^3(t)} = \frac{3 H_0^2 \Omega_{{\rm DM},0}}{8\pi G a^3}$$

and similarly for baryons too.

Decoupling the equations

- Now use the fact that $\Omega_{m,0}/\Omega_{b,0} \approx 6$ and $\delta_{\rm DM} \gtrsim \delta_b$ for scales of interest to write

$$\begin{aligned}\Omega_{\rm DM,0}\delta_{\rm DM} + \Omega_b\delta_b &= \Omega_{\rm DM,0}\delta_{\rm DM} + \Omega_{b,0}\delta_{\rm DM} - \Omega_{b,0}\delta_{\rm DM} + \Omega_{b,0}\delta_b \\ &= \Omega_{m,0}\delta_{\rm DM}\left(1 - \frac{\Omega_{b,0}}{\Omega_{m,0}} + \frac{\Omega_{b,0}\delta_b}{\Omega_{m,0}\delta_{\rm DM}}\right) \\ &\approx \Omega_{m,0}\delta_{\rm DM}\end{aligned}$$

  and hence the last equation becomes

$$\vec{\nabla}^2\phi \approx \frac{3}{2}\frac{H_0^2}{a}\Omega_{m,0}\delta_{\rm DM}$$

- With the assumption made above, one can see that the dark matter perturbations evolve independent of baryons and can be described by five equations

$$\begin{aligned}\dot{\delta}_{\rm DM} + \frac{1}{a}\,\vec{\nabla}\cdot\left[(1+\delta_{\rm DM})\,\vec{v}_{\rm DM}\right] &= 0, \\ \dot{\vec{v}}_{\rm DM} + \frac{\dot{a}}{a}\,\vec{v}_{\rm DM} + \frac{1}{a}\,(\vec{v}_{\rm DM}\cdot\vec{\nabla})\vec{v}_{\rm DM} &= -\frac{1}{a}\,\vec{\nabla}\phi, \\ \vec{\nabla}^2\phi &= \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\rm DM}}{a}.\end{aligned}$$

- These five equations contain five unknowns, namely, $\delta_{\rm DM}$, $\vec{v}_{\rm DM}$, ϕ, and hence can be solved if the initial conditions are known.

Linear dark matter perturbations

- The system of dark matter fluid equations can be solved analytically in the linear theory.
- Neglecting second order terms in perturbed quantities, our basic equations become

$$\dot{\delta}_{\rm DM} + \frac{1}{a}\,\vec{\nabla}\cdot\vec{v}_{\rm DM} = 0,$$
$$\dot{\vec{v}}_{\rm DM} + \frac{\dot{a}}{a}\,\vec{v}_{\rm DM} = -\frac{1}{a}\,\vec{\nabla}\phi,$$
$$\vec{\nabla}^2\phi = \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\rm DM}}{a}.$$

- One should note that they are identical to what we derived earlier using relativistic perturbation theory.
- From the above, one can derive a second order ordinary differential equation for $\delta_{\rm DM}$

$$\ddot{\delta}_{\rm DM} + 2\frac{\dot{a}}{a}\dot{\delta}_{\rm DM} = \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\rm DM}}{a^3}.$$

Gravitational instability

- For the moment, suppose we assume a static universe with $a = 1$. Then the solutions to the equation are

$$\delta_{\rm DM}(t) = \frac{\delta_{\rm DM}(0)}{2}\left[\exp\left(\sqrt{\frac{3H_0^2\Omega_{m,0}}{2}}\;t\right) + \exp\left(-\sqrt{\frac{3H_0^2\Omega_{m,0}}{2}}\;t\right)\right],$$

  where we have assumed $\dot{\delta}_{\rm DM}(0) = 0$.
- At late times, the contrast will grow exponentially. Thus overdense points $\delta_{\rm DM} > 0$ would become more overdense, while underdense points $\delta_{\rm DM} < 0$ would become more underdense. This is known as **gravitational instability**.

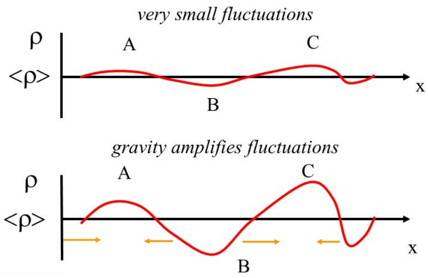

Figure taken from a talk by Michael Norman

- The presence of $\dot{a}/a$ introduces a drag term due to expansion.

Growing soutions

- The equation for the dark matter has a solution of the form $\delta_{\rm DM}(t, \vec{x}) = D(t) f(\vec{x})$, where $f(\vec{x})$ is some arbitrary function of the spatial coordinates depending on the initial configuration of the density field.
- Clearly, $D(t)$ follows the evolution equation

$$\ddot{D}(t) + 2\frac{\dot{a}}{a}\dot{D}(t) = \frac{3}{2} H_0^2 \Omega_{m,0} \frac{D(t)}{a^3}.$$

- This equation has two linearly independent solutions D_1 and D_2, of which, only one is growing with time.
- For a ΛCDM universe, the decaying solution is nothing but the Hubble parameter $D_2(a) = H(a)$, while the growing mode is given by

$$D_1(a) = H(a) \int^a \frac{\mathrm{d}a'}{H^3(a') a'^3}.$$

Growing mode in different situations

- For a matter-dominated universe, we have

$$D_2(a) = H(a) = H_0 a^{-3/2}$$

and hence

$$D_1(a) = H(a) \int^a \frac{\mathrm{d}a'}{H^3(a') a'^3} = H_0 a^{-3/2} \int^a \frac{\mathrm{d}a'}{H_0^3 a'^{-3/2}} \propto a$$

Thus the perturbations grow as the scale factor (as already seen in the relativistic perturbation theory).

- On the other hand, for a completely Λ-dominated universe, we have

$$D_2(a) = H(a) = H_0$$

and hence

$$D_1(a) \propto \int^a \frac{\mathrm{d}a'}{a'^3} \propto a^{-2}$$

It means that $D_1(a)$ is actually the decaying solution while $H =$ constant is the growing one.

- We thus write $D_1(a) = H =$ constant and $D_2(a) \propto a^{-2}$. Thus, once the universe becomes dark energy dominated, the growth of perturbations slow down and eventually stop.

Solutions to the linear equations

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- The general solution is thus given by

$$\delta_{\rm DM}(t, \vec{x}) = D_1(t) f_1(\vec{x}) + D_2(t) f_2(\vec{x}),$$

where D_1 is the growing solution and D_2 is the decaying one.
- For structure formation studies, the decaying solution is of no use as it will be dominated by the growing one at epochs of interest. Hence, from now on, by $D(t)$ we shall mean the growing solution.
- Usually, $D(t)$ is normalized such that it is unity at the present epoch.
- From the Poisson equation, we find that the potential evolves as $\phi \propto \dfrac{D}{a}$, which for a matter dominated universe becomes constant.
- From the continuity equation, we see that the peculiar velocity evolves as $\vec{v}_{\rm DM} \propto a\, \dot{D}$.
- Conventionally, the linear evolution of the peculiar velocity field is written as

$$\vec{v}_{\rm DM}(a) \propto a\, D(a)\, H(a)\, f(a), \qquad f(a) \equiv \frac{\dot{D}}{D}\,\frac{a}{\dot{a}} = \frac{{\rm d} \ln D(a)}{{\rm d} \ln a}.$$

- For calculational purposes, $f(a)$ or $f(z)$ can be well approximated by a fitting function of the form

$$f(z) \approx \Omega_{m,0}^{4/7}(z) = \left[\frac{\Omega_{m,0}(1+z)^3}{H^2(z)/H_0^2}\right]^{4/7}.$$

- Note that $f(z)$ is very close to unity at redshifts $z > 2$ for flat cosmological models.

Baryonic perturbations

- Let us now turn our attention to the baryonic equations

$$\dot{\delta}_b + \frac{1}{a}\,\vec{\nabla}\cdot\left[\left(1+\delta_b\right)\,\vec{v}_b\right] = 0,$$

$$\dot{\vec{v}}_b + \frac{\dot{a}}{a}\,\vec{v}_b + \frac{1}{a}\,(\vec{v}_b\cdot\vec{\nabla})\vec{v}_b = -\frac{1}{a}\,\vec{\nabla}\phi - \frac{\vec{\nabla}p_b}{a\bar{\rho}_b(1+\delta_b)},$$

  where we assume that ϕ is already obtained by solving the dark matter evolution equations.
- Note that the above system has five unknown variables, namely, δ_b, $\vec{v}_b$, p_b but only four equations.
- Hence, to solve the system, one needs to provide a relation between the density and pressure of the baryons, loosely called the "effective the equation of state".
- One way to address this is by specifying the sound speed

$$c_s^2 \equiv \frac{\partial p_b}{\partial \rho_b} \quad\Longrightarrow\quad \vec{\nabla}p_b = c_s^2\vec{\nabla}\rho_b = c_s^2\,\bar{\rho}_b\vec{\nabla}\delta_b.$$

- Hence the Euler equation becomes

$$\dot{\vec{v}}_b + \frac{\dot{a}}{a}\,\vec{v}_b + \frac{1}{a}\,(\vec{v}_b\cdot\vec{\nabla})\vec{v}_b = -\frac{1}{a}\,\vec{\nabla}\phi - c_s^2\,\frac{\vec{\nabla}\delta_b}{a(1+\delta_b)}.$$

Linear baryonic perturbations

- The system of fluid equations for baryons too can be solved exactly in the linear theory.
- Neglecting second order terms in perturbed quantities, our basic equations become

$$\dot{\delta}_b + \frac{1}{a}\,\vec{\nabla}\cdot\vec{v}_b = 0,$$

$$\dot{\vec{v}}_b + \frac{\dot{a}}{a}\,\vec{v}_b = -\frac{1}{a}\,\vec{\nabla}\phi - \frac{c_s^2}{a}\vec{\nabla}\delta_b.$$

- The evolution equation for δ_b is

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b - \frac{c_s^2}{a^2}\vec{\nabla}^2\delta_b = \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\rm DM}}{a^3}.$$

Fourier solutions

- To obtain the linear solutions, it is more convenient to work in the Fourier domain (for both DM and b)

$$\delta(\vec{k}) = \int \mathrm{d}^3x\, \delta(\vec{x}) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}, \quad \delta(\vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \delta(\vec{k}) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}},$$

  and similarly for other quantities.
- Clearly, the Fourier transform of $\vec{\nabla}\delta(\vec{x})$ is $\mathrm{i}\vec{k}\delta(\vec{k})$ and that of $\vec{\nabla}^2\delta(\vec{x})$ is $-k^2\delta(\vec{k})$.
- Then the equation for δ_b in Fourier space turns out to be

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b + k^2\frac{c_s^2}{a^2}\delta_b = \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\mathrm{DM}}}{a^3}.$$

- Note that according to the linear theory, the Fourier modes $\delta(\vec{k}, t)$ evolve independent of each other.
- At this point, define a new quantity known as the **Jeans length** or **Jeans scale**

$$x_J \equiv \frac{c_s}{H_0}\sqrt{\frac{2a}{3\Omega_{m,0}}}.$$

- In terms of sound speed, one can see that $x_J \sim c_s/\sqrt{G\rho_m}$.
- The equation then takes the form

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b + \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_b}{a^3}\,(x_J^2 k^2) = \frac{3}{2}H_0^2\Omega_{m,0}\frac{\delta_{\mathrm{DM}}}{a^3}.$$

Solutions to the baryonic perturbations

- In the simple situation where x_J is independent of time, the solution of the above equation is

$$\delta_b(t, \vec{k}) = \frac{\delta_{\rm DM}(t, \vec{k})}{1 + x_J^2 k^2}.$$

- The above equation shows that at scales much larger than x_J, i.e., for $k \ll x_J^{-1}$, we have $\delta_b \approx \delta_{\rm DM}$. Thus the baryon and dark matter evolve identically. This is expected because the pressure does not play any role at very large scales.
- On smaller scales, we find $\delta_b \approx \delta_{\rm DM}/(x_J^2 k^2)$, showing that the perturbations in baryons are suppressed because of pressure support.
- Using the linearity of equations, one can show that the baryonic velocity field evolves as

$$\vec{v}_b = \frac{\vec{v}_{\rm DM}}{1 + x_J^2 k^2}.$$

Evolution of Jeans scale

- The evolution of Jeans scale depends on the evolution of the baryon (gas) temperature.
- For an ideal gas, we can write
$$p_b = \frac{\rho_b k_B T}{\mu m_p},$$
where $\mu \equiv \rho_b / \mu n_b$ is the mean molecular weight.
- If we assume that
$$T = T_0 \left(\frac{\rho_b}{\bar{\rho}_b}\right)^{\gamma - 1},$$
valid for low-density gas in the intergalactic medium, then
$$c_s^2 = \gamma \frac{k_B T}{\mu m_p}.$$
- The Jeans scale is
$$x_J = \frac{1}{H_0} \sqrt{\frac{2 a \gamma k_B T}{3 \mu m_p \Omega_{m,0}}}.$$
- Typical value is $x_J \sim 100$ kpc (comoving) at $z \sim 3$ (assuming $T \sim 10^4$ K).

Evolution of gas temperature

- In absence of any interaction, we expect $T \propto a^{-2} \propto (1+z)^2$. In general, it is determined by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -2\frac{\dot{a}}{a}T + \frac{x_e(t)}{t_T}\frac{T_r - T}{a^4} + \frac{2}{3k_B n_b}\mathcal{H},$$

where $t_T \equiv \frac{3m_e}{8\bar{\rho}_{r,0}\sigma_T c}$, $T_r = 2.73\ \mathrm{K}/a$ and the three terms on the right are
  - adiabatic cooling $\propto (1+z)^2$,
  - Thomson scattering off free electrons left-over from recombination,
  - net heating arising from structure formation / galaxy formation / reionization.
- At early times $T \propto (1+z)$, same as radiation. At $z \sim 200$, it decouples from radiation and $T \propto (1+z)^2$. The gas heats up once star formation begins.