Cosmology
Lecture 7

Inflation, scalar fields and dark energy

Tirthankar Roy Choudhury
National Centre for Radio Astrophysics
Tata Institute of Fundamental Research
Pune

NCRA • TIFR

Solution of the horizon problem

- We have seen that the horizon size at, say, the recombination epoch (or the last scattering surface) is too small to explain the isotropy of the CMB.
- We have also seen that the horizon size can be made arbitrarily large if we allow $a(t) \propto t^\alpha$ with $\alpha > 1$. Note that this implies

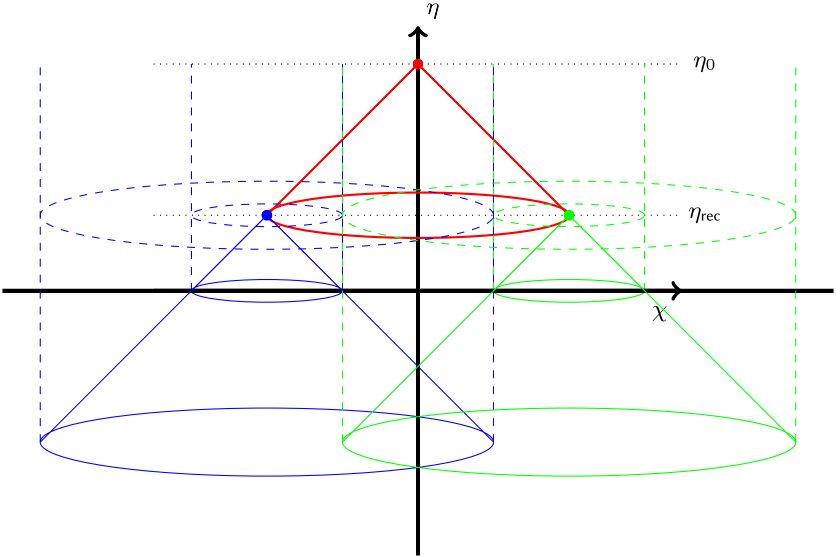
$$\ddot{a} \propto \alpha(\alpha - 1)t^{\alpha-2} > 0,$$

  i.e., the universe must accelerate.
- Invoking a phase of accelerated expansion, where the universe expands rapidly, can solve the horizon problem. Such a accelerating phase is known as **inflation**.
- This can also be understood using the conformal time

$$\eta = \int \frac{\mathrm{d}t}{a(t)} = (1-\alpha)\ t_0^\alpha\ t^{1-\alpha} = (1-\alpha)\ t_0\ a^{(1-\alpha)/\alpha}.$$

- When $\alpha < 1$, we have $\eta \to 0$ as $a \to 0$. Thus, for a radiation dominated universe, the conformal time coordinate begins from $\eta = 0$ at the Big Bang.
- However, when $\alpha > 1$, we find that $\eta \to -\infty$ when $a \to 0$. Thus the lower bound on η can be infinitely large negative.

Inflation and the horizon problem



Inflation and the flatness problem

- The flatness problem can also be solved within the inflationary universe.
- Note that for $a(t) \propto t^{\alpha}$, we get
$$H(a) = \frac{\dot{a}}{a} \propto a^{-1/\alpha}.$$
- Let a_i and a_f denote the scale factors at the start and end of inflation, respectively. Now we have already shown that $\Omega_k(a) = H_0^2\, \Omega_{k,0}/[H^2(a)\, a^2]$, and hence
$$\frac{\Omega_k(a_f)}{\Omega_k(a_i)} = \frac{H^2(a_i)a_i^2}{H^2(a_f)a_f^2} \approx \left(\frac{a_i}{a_f}\right)^{2-2/\alpha}.$$
- Now, during inflation, we expect the scale factor to increase by a large amount $a_f \ll a_i$. If $\alpha > 1$, we have $2 - 2/\alpha > 0$. Hence
$$\frac{\Omega_k(a_f)}{\Omega_k(a_i)} \ll 1,$$
showing that the universe becomes highly flat at the end of inflation *irrespective of its initial curvature*. This solves the flatness problem.

Some details on inflation

- From the amplitude of density fluctuations observed, one expects the energy scale of inflation to be $\sim 10^{16}$ GeV, which is $\sim 10^{-35}$ s after Big Bang.
- This energy scale corresponds to the Grand Unified Theory energy (the energy above which the electromagnetic, weak and strong forces unify). For reference, the Planck energy scale is $\sim 10^{19}$ GeV.
- The scale factor should increase by a factor $\sim \mathrm{e}^{60} \sim 10^{26}$ during inflation to solve the horizon and flatness problems. The increase is measured by the number of e-folds.
- After exiting from inflation, the universe evolves as in the radiation dominated era.

Scalar field Lagrangian

- At very early times, it is not possible to have fluid-like matter. The most natural way to describe energy content in the universe suring inflation is using (quantum) field theory.
- In field theory, one starts from a Lagrangian for the field and obtains the equation of motion. The Lagrangian (or Hamiltonian) can be used to quantize the field if required.
- For example, the Lagrangian for the electromagnetic field, described by A^i, is $\mathcal{L} = -\frac{1}{16\pi} F^{ik} F_{ik} - A_i \, j^i$.
- Similarly, the Lagrangian for the Dirac particles is $\mathcal{L} = \mathrm{i}\hbar \, \bar{\psi} \, \gamma^k \, \psi_{;k} - m \bar{\psi} \psi, \; \bar{\psi} = \psi^\dagger \, \gamma^0$.
- The simplest way to achieve inflation is via a scalar field $\Phi(x^i)$ having the Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{ik} \, \Phi_{;i} \, \Phi_{;k} - V(\Phi),$$

  where $V(\Phi)$ is the potential under which the field evolves.
- The scalar field Φ that drives inflation is called the **inflaton**.
- The Euler-Lagrange equations are given by

$$\left(\frac{\partial \mathcal{L}}{\partial(\Phi_{;i})} \right)_{;i} = \frac{\partial \mathcal{L}}{\partial \Phi} \Longrightarrow g^{ik} \, \Phi_{;ik} + \frac{\mathrm{d} V(\Phi)}{\mathrm{d}\Phi} = 0.$$

Cosmological scalar fields

- At large scales, the scalar field must be homogeneous $\Phi = \Phi(t)$.
- The Lagrangian resembles that of a particle in a potential

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}^2 - V(\Phi).$$

- In this case $g^{ik}\,\Phi_{;ik} = \ddot{\Phi} + 3\frac{\dot{a}}{a}\dot{\Phi}$, hence the equation of motion is

$$\ddot{\Phi} + 3\frac{\dot{a}}{a}\dot{\Phi} + \frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi} = 0.$$

  This is nothing but the **Klein-Gordon equation** in an expanding universe.
- The effect of expansion, i.e., the $\dot{a}/a$ term acts as a "damping" in the force equation.
- The equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right)a^3\right] = -\left(\frac{1}{2}\dot{\Phi}^2 - V(\Phi)\right)\frac{\mathrm{d}(a^3)}{\mathrm{d}t},$$

  which has the form $\mathrm{d}(\rho a^3) = -P\,\mathrm{d}(a^3)$.
- We can thus define an effective density and pressure for the scalar field

$$\rho_\Phi = \frac{1}{2}\dot{\Phi}^2 + V(\Phi), \quad P_\Phi = \frac{1}{2}\dot{\Phi}^2 - V(\Phi).$$

Inflation driven by scalar fields

- The effective equation of state for the scalar field is

$$w_\Phi = \frac{\dot{\Phi}^2/2 - V(\phi)}{\dot{\Phi}^2/2 + V(\phi)}.$$

- We have already seen the condition for inflation is accelerated expansion. This implies $w < -1/3$, hence we must have

$$\dot{\Phi}^2 < V(\Phi).$$

- The Friedmann equations for a universe with a scalar field are

$$H^2 = \frac{8\pi G}{3}\rho_\Phi = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\right], \quad \dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -4\pi G(\rho_\Phi + P_\Phi) = -4\pi G\,\dot{\Phi}^2.$$

- These equations can be inverted to give

$$\Phi = \frac{1}{\sqrt{4\pi G}}\int \mathrm{d}t\,\sqrt{-\dot{H}}, \quad V = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\Phi}^2 = \frac{1}{8\pi G}\left(3H^2 + \dot{H}\right).$$

Given a form for $a(t)$, the above can be solved to obtain $V(\Phi)$ parametrically in terms of t.

Slow roll approximation

- In the extreme case when $w \to -1$, we have exponential expansion. In that case $\dot{\Phi}^2 \ll V(\Phi)$, i.e., the kinetic energy of the field must be small.
- In case we want $\dot{\Phi}^2$ to remain small for a long period of time, we also need $\ddot{\Phi}$ to be small.
- The Klein-Gordon equation then implies that $V'(\Phi) \approx 0$, i.e., the potential must be flat and the field must be rolling down slowly in it. This is known as the **slow roll approximation**.
- Assuming $\ddot{\Phi} \to 0$ in the Klein-Gordon equation, we get

$$3H\dot{\Phi} \approx -V'(\Phi),$$

while putting $\dot{\Phi}^2 \ll V(\Phi)$ in the Friedmann equation gives

$$3H^2 \approx 8\pi G\, V(\Phi).$$

- These two equations give

$$\frac{\dot{\Phi}}{H} \approx -\frac{1}{8\pi G}\frac{V'(\Phi)}{V(\Phi)} \implies \frac{\dot{\Phi}}{\sqrt{8\pi G/3}\,\sqrt{V(\Phi)}} \approx \frac{1}{8\pi G}\frac{V'(\Phi)}{V(\Phi)} \implies \frac{3\dot{\Phi}^2}{V(\Phi)} \approx \frac{1}{8\pi G}\left[\frac{V'(\Phi)}{V(\Phi)}\right]^2$$

and the slow roll condition gives

$$\frac{1}{8\pi G}\left[\frac{V'(\Phi)}{V(\Phi)}\right]^2 \ll 1.$$

Slow roll parameters

- Let us next find out the implication of neglecting $\ddot{\Phi}$ in the Klein-Gordon equation.
- Using $\dot{\Phi} \approx -V'(\Phi)/3H$, we get

$$\ddot{\Phi} \approx -\frac{V''(\Phi)}{3H}\dot{\Phi} + \frac{V'(\Phi)}{3H^2}\dot{H} \approx \frac{V''(\Phi)}{V'(\Phi)}\dot{\Phi}^2 + \frac{V'(\Phi)}{8\pi G V(\Phi)} \times \left(-4\pi G \dot{\Phi}^2\right) \approx \left[\frac{V''(\Phi)}{V'(\Phi)} - \frac{1}{2}\frac{V'(\Phi)}{V(\Phi)}\right]\dot{\Phi}^2.$$

- Hence

$$\frac{\ddot{\Phi}}{H\dot{\Phi}} \approx \left[\frac{V''(\Phi)}{V'(\Phi)} - \frac{1}{2}\frac{V'(\Phi)}{V(\Phi)}\right]\frac{\dot{\Phi}}{H} \approx -\left[\frac{V''(\Phi)}{V'(\Phi)} - \frac{1}{2}\frac{V'(\Phi)}{V(\Phi)}\right]\frac{1}{8\pi G}\frac{V'(\Phi)}{V(\Phi)}$$
$$\approx -\left[\frac{1}{8\pi G}\frac{V''(\Phi)}{V(\Phi)} - \frac{1}{16\pi G}\left(\frac{V'(\Phi)}{V(\Phi)}\right)^2\right].$$

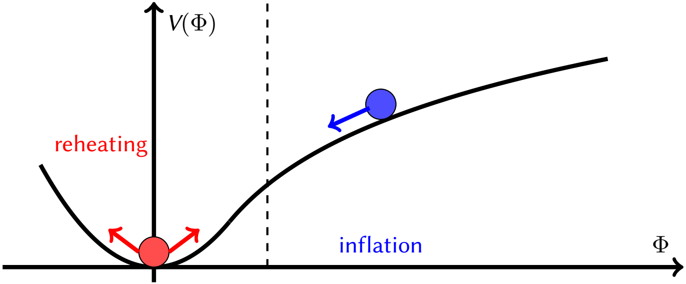
- So, in addition to the previous condition $(8\pi G)^{-1}(V'/V)^2 \ll 1$, we must also have $(8\pi G)^{-1}(V''//V) \ll 1$ for the slow roll to persist.
- It is customary to define two slow roll parameters

$$\epsilon = \frac{1}{16\pi G}\left[\frac{V'(\Phi)}{V(\Phi)}\right]^2, \quad \eta = \frac{1}{8\pi G}\frac{V''(\Phi)}{V(\Phi)}$$

and these parameters must be $\ll 1$ for inflation to be effective.

Exit from inflation

- The end of inflation happens when the slow roll conditions do not hold any more. This could be because the field reached the minimum of the potential and oscillates rapidly.



- If the scalar inflaton field is coupled to other fields (e.g., photons), then the inflaton energy can be converted into radiation and the universe enters the radiation dominated phase. This process is known as **reheating**.
- Although the mechanism seems to work conceptually, the details are quite complex and still to be worked out fully.

Cosmological constant and the vacuum energy

- We have already seen that the Einstein equation, in the presence of Λ, becomes

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G \left(T_{ik} + \frac{\Lambda}{8\pi G} g_{ik} \right).$$

The Λ-term can be interpreted as the **vacuum energy**.

- In this interpretation, the vacuum is believed to have a stress-energy tensor

$$T^{(\mathrm{vac})}{}_{ik} = \frac{\Lambda}{8\pi G} g_{ik} \Longrightarrow T^{(\mathrm{vac})}{}^{i}{}_{k} = \frac{\Lambda}{8\pi G} \delta^{i}{}_{k} \Longrightarrow P_{\mathrm{vac}} = -\rho_{\mathrm{vac}} = -\frac{\Lambda}{8\pi G}.$$

- Interestingly, if we demand that the vacuum stress tensor must be the same all inertial observers (otherwise we can define an absolute reference frame), we can compute the equation of state of the vacuum energy.
- In the vacuum rest frame, we have $T^{(\mathrm{vac})}{}^{0}{}_{0} = \rho_{\mathrm{vac}}$, $T^{(\mathrm{vac})}{}^{\alpha}{}_{\alpha} = -P_{\mathrm{vac}}$. Consider another frame moving with a speed v along the x-direction, then

$$t' = \gamma(t - vx), \;\; x' = \gamma(x - vt) \Longrightarrow t = \gamma(t' + vx'), \;\; x = \gamma(x' + vt').$$

- In the primed frame, let us calculate, say,

$$T'^{(\mathrm{vac})}{}^{0}{}_{1} = \frac{\partial x'^{0}}{\partial x^{m}} \frac{\partial x^{n}}{\partial x'^{1}} T^{(\mathrm{vac})}{}^{m}{}_{n} = \gamma^{2} v (\rho_{\mathrm{vac}} + P_{\mathrm{vac}}).$$

- The principle of special relativity implies that this component must vanish, hence we must have $P_{\mathrm{vac}} = -\rho_{\mathrm{vac}}$.

Infinite vacuum energy

- Now, in absence of everything else, the vacuum energy would consist of quantum gravitational fluctuations

$$E_{\rm vac} = \sum_{\vec{k}} \frac{1}{2}\hbar\omega_{\vec{k}} = V \int \frac{{\rm d}^3 k}{(2\pi)^3} \frac{1}{2}\hbar\omega_{\vec{k}}.$$

- Writing $\hbar\omega_{\vec{k}} = \sqrt{\vec{p}^2 + m^2} = \sqrt{\hbar^2 \vec{k}^2 + m^2}$, we find the energy density to be

$$\begin{aligned}
\rho_{\rm vac} = \frac{E_{\rm vac}}{V} &= \frac{1}{16\pi^3} \int {\rm d}^3 k \sqrt{\hbar^2 \vec{k}^2 + m^2} = \frac{1}{4\pi^2} \int_0^{k_{\rm max}} {\rm d}k\, k^2 \sqrt{\hbar^2 k^2 + m^2} \\
&= \frac{1}{4\pi^2} \left[\int_0^{k_1} {\rm d}k\, k^2 \sqrt{\hbar^2 k^2 + m^2} + \int_{k_1}^{k_{\rm max}} {\rm d}k\, k^2 \sqrt{\hbar^2 k^2 + m^2}\right] \qquad \left(k_1 \gg \frac{m}{\hbar} \longrightarrow \frac{mc}{\hbar}\right) \\
&\approx \frac{1}{4\pi^2} \left[I(k_1) + \hbar \int_{k_1}^{k_{\rm max}} {\rm d}k\, k^3\right] \approx \frac{1}{4\pi^2} \left[I(k_1) + \frac{\hbar}{4}(k_{\rm max}^4 - k_1^4)\right] \\
&\approx \frac{\hbar}{16\pi^2} k_{\rm max}^4 \longrightarrow \frac{\hbar c}{16\pi^2} k_{\rm max}^4,
\end{aligned}$$

where we have taken $k_{\rm max} \gg k_1$.

- Clearly the vacuum energy diverges unless the integral is cut-off at small scales (high energies).

Magnitude of the vacuum energy

- The natural cut-off would the Planck scale $L_{\rm Pl}$ where the conventional quantum field theory breaks down and quantum gravitational effects could be important. Then, we take $k_{\rm max} \sim L_{\rm Pl}^{-1}$, so that

$$\rho_{\rm vac} \sim \frac{\hbar c}{L_{\rm Pl}^4}.$$

- Now, $L_{\rm Pl}^2 = \hbar G / c^3$, hence

$$\rho_{\rm vac} \sim \frac{\hbar c^7}{\hbar^2 G^2} = \frac{c^7}{\hbar G^2} \approx 4.6 \times 10^{114} \text{ erg cm}^{-3},$$

  which is the *Planck energy density*. The corresponding mass density is

$$\rho_{\rm vac} \sim \frac{c^5}{\hbar G^2} \approx 5.1 \times 10^{93} \text{ gm cm}^{-3}.$$

- The energy density corresponding to the cosmological constant is

$$\rho_\Lambda \sim \rho_{c,0} \sim 10^{-29} \text{ gm cm}^{-3}.$$

- Hence

$$\frac{\rho_\Lambda}{\rho_{\rm vac}} \sim 10^{-122},$$

  which is several order of magnitudes smaller than expected.
- This is known as the **small cosmological constant problem**.

Cosmological constant as curvature

- A different interpretation of the cosmological constant is to write the Einstein equation as

$$R_{ik} - \frac{1}{2} g_{ik} R - \Lambda g_{ik} = 8\pi G T_{ik}.$$

- In this picture, Λ can be related to the curvature of the vacuum. If $T_{ik} = 0$, we have

$$g^{ik} R_{ik} = R = -4\Lambda.$$

- The value of Λ, even in this case, needs to be fixed from observations and there is no natural way to determine its value.

The coincidence problem

- There is another problem related to the cosmological constant known as the **coincidence problem**.
- It turns out that we live in a special epoch when $\Omega_\Lambda \sim \Omega_{m,0}$. For any other epoch
$$\frac{\Omega_\Lambda(a)}{\Omega_m(a)} = \frac{\Omega_\Lambda a^3}{\Omega_{m,0}} \sim a^3,$$
which shows that $\Omega_\Lambda \ll \Omega_m(a)$ at early epochs (and vice versa in the future).
- People have invoked the concept of **dark energy** which has an evolving $\Omega_{\rm DE}(a)$ such that the coincidence problem does not arise.
- Consider a matter component having a time-varying equation of state, i.e., $P_{\rm DE}(a) = w_{\rm DE}(a)\rho_{\rm DE}(a)$.
- Now use the energy conservation equation
$$\frac{{\rm d}\rho_{\rm DE}}{{\rm d}t} = -3\frac{\dot{a}}{a}(\rho_{\rm DE} + P_{\rm DE}) = -3\frac{\dot{a}}{a}\left[1 + w_{\rm DE}(t)\right]\rho(t) \implies \frac{{\rm d}\rho_{\rm DE}}{{\rm d}a} = -\frac{3}{a}\left[1 + w_{\rm DE}(a)\right]\rho_{\rm DE}(a),$$
to obtain
$$\rho_{\rm DE}(a) = \frac{\rho_{\rm DE,0}}{a^3}\exp\left[3\int_a^1 \frac{{\rm d}a'}{a'} w_{\rm DE}(a')\right].$$
- A careful choice of $w_{\rm DE}(a)$ might allow us to avoid the coincidence problem.
- A natural way to obtain evolving dark energy is to use a scalar field rolling down a potential. Such models are known as **quintessence**.