Cosmology
Lecture 5
FLRW dynamics

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The Einstein tensor

- Till now, we have studying the properties of the FLRW metric without solving for the dynamics.
  The Einstein equation would determine the dependence of the expansion on the components of the universe.
- Let us use the form of the metric as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right].$$

  We will use $c = 1$ again and put back factors of c when needed.
- The components of Einstein tensor are

$$G^0_{\ 0} = g^{00} G_{00} = 3\frac{k + \dot{R}^2}{R^2}$$
$$G^1_{\ 1} = g^{11} G_{11} = \frac{k + \dot{R}^2 + 2R\ddot{R}}{R^2}$$
$$G^2_{\ 2} = g^{22} G_{22} = \frac{k + \dot{R}^2 + 2R\ddot{R}}{R^2}$$
$$G^3_{\ 3} = g^{33} G_{33} = \frac{k + \dot{R}^2 + 2R\ddot{R}}{R^2}.$$

- Also recall that the stress-energy tensor is (using $c = 1$)

$$T^i_{\ j} = \mathrm{diag}(\rho, -P, -P, -P).$$

Friedmann equations

- The ${}^0{}_0$ component of Einstein equation $G^i{}_j = 8\pi G\, T^i{}_j$ turns out to be

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho.$$

- The ${}^\alpha{}_\alpha$ component gives

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} + 2\frac{\ddot{R}}{R} = -8\pi G P,$$

which, when combined with the previous equation, becomes

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3P).$$

- We find that the acceleration is produced, not by the energy density alone, but by the combination $\rho + 3P$.
- The above differential equations are called **Friedmann equations**.
- The two Friedmann equations can be combined to give the energy conservation equation

$$\dot{\rho} = -3\frac{\dot{R}}{R}\left(\rho + P\right).$$

This is a confirmation of the fact that the equations of motion are contained within the Einstein equation.

Critical density and density parameter

- If the universe is flat $k = 0$, then the expansion rate is related to the density by

$$H^2(t) = \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho(t).$$

- Thus, the flat universe has a very particular density, which called the **critical density**:

$$\rho_c(t) \equiv \frac{3H^2(t)}{8\pi G}.$$

- The value of ρ_c at present is given by $\rho_{c,0} = 3H_0^2/8\pi G = 1.88 \times 10^{-29} h^2$ gm cm^{-3} $= 2.78 \times 10^{11} h^2 M_\odot \mathrm{Mpc}^{-3}$.
- A universe which is **spatially closed** ($k = +1$) will have

$$H^2(t) = \frac{8\pi G}{3}\rho(t) - \frac{k}{R^2} < \frac{8\pi G}{3}\rho(t) \quad \Longrightarrow \quad \rho(t) > \rho_c(t).$$

- A **spatially open** ($k = -1$) will have $\rho(t) < \rho_c(t)$.
- The **density parameter** is defined as

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\,\rho(t)}{3H^2(t)}.$$

  The value at the present epoch is denoted by Ω_0 or simply Ω.
- Thus, a closed universe ($k > 0$) corresponds to $\Omega(t) > 1$, while an open universe ($k < 0$) corresponds to $\Omega(t) < 1$. The $\Omega(t) = 1$ universe is spatially flat.

The present value of the scale factor

- The value of $R_0 \equiv R(t_0)$ for $k \neq 0$ universe can be obtained by evaluating the first Friedmann equation at the present epoch t_0

$$H_0^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_0 = H_0^2 \frac{\rho_0}{\rho_{c,0}} = H_0^2 \Omega_0 \Longrightarrow R_0^2 = \frac{k}{H_0^2(\Omega_0 - 1)},$$

  and hence

$$R_0 = \frac{1}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}}.$$

- We can define a parameter which is the "effective density of the curvature":

$$\Omega_k(t) \equiv 1 - \Omega(t).$$

- The value of R_0 becomes

$$R_0 = \frac{1}{H_0}\sqrt{\frac{-k}{\Omega_{k,0}}} = \frac{1}{H_0\sqrt{|\Omega_{k,0}|}}.$$

- When $k = 0$, the value of R_0 cannot be determined. However, it turns out that all the observables are independent of R_0 for the flat universe.

The normalized scale factor

- We can define a **normalized scale factor**
$$a(t) \equiv \frac{R(t)}{R_0},$$
so that its present value is unity.
- Note that $H(t) = \dot{a}/a$.
- In terms of a, the metric becomes
$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t) R_0^2 \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2\right] = \mathrm{d}t^2 - a^2(t) \left[\frac{\mathrm{d}r'^2}{1 - kr'^2/R_0^2} + r'^2 \mathrm{d}\Omega^2\right],$$
where $r' = R_0 r$. For the other form
$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t) R_0^2 \left[\mathrm{d}\chi^2 + S_k^2(\chi) \mathrm{d}\Omega^2\right] = \mathrm{d}t^2 - a^2(t) \left[\mathrm{d}\chi'^2 + R_0^2 S_k^2(\chi'/R_0) \mathrm{d}\Omega^2\right],$$
where $\chi' = R_0 \chi$.
- It is interesting to note that R_0 cancels out from the expressions only when $k = 0$, hence the value of R_0 has no relevance in the flat universe.
- All the expressions we have obtained so far can be re-written in terms of $a(t)$. For example,
$$1 + z = \frac{1}{a(t)}.$$

General solutions to the Friedmann equations

- Consider a universe with various *non-interacting* components with equation of state $P_\alpha = w_\alpha \rho_\alpha$.
- Each of these components evolve as $\rho_\alpha \propto a^{-3(1+w_\alpha)}$, or

$$\rho_\alpha(t) = \rho_{\alpha,0}\ a^{-3(1+w_\alpha)}.$$

- Then the first Friedmann equation gives

$$\begin{aligned} H^2(a) &= \frac{8\pi G}{3} \sum_\alpha \rho_\alpha - \frac{k}{R_0^2 a^2} \\ &= \frac{8\pi G}{3} \sum_\alpha \rho_{\alpha,0}\ a^{-3(1+w_\alpha)} - \frac{k}{R_0^2 a^2} \\ &= H_0^2 \sum_\alpha \frac{\Omega_{\alpha,0}}{a^{3(1+w_\alpha)}} - \frac{k}{R_0^2 a^2}. \end{aligned}$$

- At the present epoch $t = t_0, a = 1$, we get

$$\frac{k}{R_0^2} = H_0^2 \left(\sum_\alpha \Omega_{\alpha,0} - 1 \right) = -H_0^2 \Omega_{k,0}, \quad \Omega_{k,0} \equiv 1 - \sum_\alpha \Omega_{\alpha,0} = 1 - \Omega_{\text{tot},0}.$$

- The Hubble parameter is then given by

$$H^2(a) = H_0^2 \left[\sum_\alpha \frac{\Omega_{\alpha,0}}{a^{3(1+w_\alpha)}} + \frac{\Omega_{k,0}}{a^2} \right].$$

General solutions to the Friedmann equations (contd)

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- $$H^2(a) = \frac{\dot{a}^2}{a^2} = H_0^2 \left[\sum_\alpha \frac{\Omega_{\alpha,0}}{a^{3(1+w_\alpha)}} + \frac{\Omega_{k,0}}{a^2} \right].$$
- In principle, this can be solved to obtain $a(t)$ provided we know all the source components present in the Universe.
- Also, given $H(a)$ or equivalently

  $$H^2(z) = H_0^2 \left[\sum_\alpha \Omega_{\alpha,0} \, (1+z)^{3(1+w_\alpha)} + \Omega_{k,0} \, (1+z)^2 \right],$$

  we can calculate all observables like the distances etc.
- Note that curvature acts as a component having density parameter Ω_k and equation of state $w_k = -1/3$.
- The density parameter for any of the components β at any epoch a can be written as

$$\Omega_\beta(a) = \frac{\rho_\beta(a)}{\rho_c(a)} = \frac{8\pi G \rho_\beta(a)}{3H^2(a)} = \frac{8\pi G \rho_{\beta,0} \, a^{-3(1+w_\beta)}}{3H_0^2 \left(\sum_\alpha \Omega_{\alpha,0} a^{-3(1+w_\alpha)} + \Omega_{k,0}/a^2 \right)}$$
$$= \frac{\Omega_{\beta,0} \, a^{-3(1+w_\beta)}}{\sum_\alpha \Omega_{\alpha,0} a^{-3(1+w_\alpha)} + \Omega_{k,0}/a^2}.$$

Flat single-component universe

- Consider the case where the universe is flat ($k = 0$) and is filled with only one kind of matter which has an equation of state $P = w\rho$. This can happen, e.g., when one of the components dominates over all others.
- Then the density evolve as

$$\rho(t) = \rho_0 \, a^{-3(1+w)}.$$

  Also $\Omega(t) = 1$ because $k = 0$.
- Then the solution can be obtained from

$$H^2(t) = \frac{\dot{a}^2}{a^2} = H_0^2 a^{-3(1+w)}.$$

- For $w > -1$, this can be solved as

$$\dot{a} = H_0 \, a^{-(1+3w)/2} \Longrightarrow a = \left[\frac{3(1+w)H_0 t}{2}\right]^{2/[3(1+w)]},$$

  where the constant of integration can be fixed by choosing $a = 0$ at $t = 0$.
- For $w = -1$, we have $a(t) \propto \mathrm{e}^{H_0 t}$.
- For matter-dominated universe $w = 0$, and hence

$$a \propto t^{2/3},$$

  and for radiation-dominated universe $w = 1/3$, and hence

$$a \propto t^{1/2}.$$

- A flat matter-dominated universe is called **Einstein-deSitter universe**.

Properties of the flat single-component universe with $w > -1$

- The age of the universe can be obtained in terms of the Hubble parameter as
$$t = \frac{2}{3(1+w)} H^{-1}(t).$$
- The age of the universe can also be related to the density as
$$t = \frac{1}{H_0} \frac{2a^{3(1+w)/2}}{3(1+w)} = \frac{1}{H_0} \frac{2}{3(1+w)} \left(\frac{\rho_0}{\rho}\right)^{1/2} = \frac{2}{3(1+w)} \left(\frac{3}{8\pi G \rho_0}\right)^{1/2} \left(\frac{\rho_0}{\rho}\right)^{1/2},$$
which leads to
$$\rho(t) = \frac{1}{6(1+w)^2 \pi G t^2}.$$
Interestingly $\rho \propto t^{-2}$, independent of w.
- The acceleration is given by
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\rho = -\frac{H_0^2}{2}(1+3w)\frac{\rho}{\rho_0} = -\frac{H_0^2}{2}(1+3w) a^{-3(1+w)}.$$
- The acceleration is negative for matter and radiation, the universe decelerates when filled with normal matter.
- The deceleration parameter is given by
$$q(t) = -\frac{\ddot{a}\, a}{\dot{a}^2} = -\frac{\ddot{a}}{a} \frac{a^2}{\dot{a}^2} = \frac{H_0^2}{2}(1+3w) a^{-3(1+w)} \times \frac{1}{H_0^2} a^{3(1+w)} = \frac{1+3w}{2}.$$
This is independent of t and is > 0 for $w > -1/3$. Thus the universe can accelerate only when $w < -1/3$.

Matter-dominated Universe

- The next simple solution is for a matter-dominated universe ($w = 0$), but with $k \neq 0$.
- Then

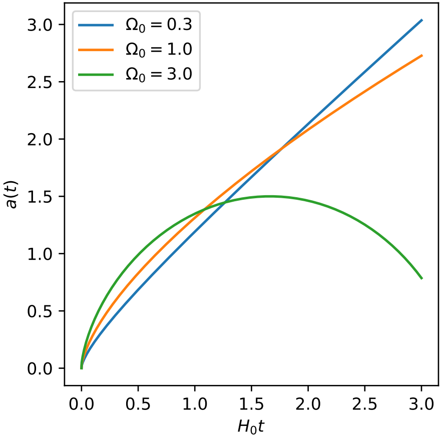
$$H^2 = H_0^2 \left[\frac{\Omega_0}{a^3} + \frac{\Omega_{k,0}}{a^2}\right] = H_0^2 \left[\frac{\Omega_0}{a^3} + \frac{1 - \Omega_0}{a^2}\right].$$

- Then, the equation to be solved will be

$$\dot{a}^2 = \frac{H_0^2 \Omega_0}{a} + H_0^2 \Omega_{k,0} \quad \Longrightarrow \quad \frac{1}{2}\dot{a}^2 - \frac{H_0^2 \Omega_0}{2a} = \frac{H_0^2 \Omega_{k,0}}{2}.$$

- This is like the energy equation for a particle moving in the r^{-2} gravitational field. The quantity $H_0^2 \Omega_{k,0}/2$ plays the role of the total conserved energy.
- When $\Omega_{k,0} = 1 - \Omega_0 > 0$, the motion is unbounded. Hence the universe will expand forever when $\Omega_0 < 1$ (i.e., there is not enough matter to halt the expansion).
- The opposite will happen for $\Omega_0 > 1$ where it will expand followed by a contraction phase. Too much matter makes the universe recollapse.
- The flat universe corresponds to the case when the particle moves with the escape velocity, i.e., the universe will keep on expanding asymptotically.

Evolution in matter-dominated models



- We have already found the solution for $\Omega_0 = 1$ (Einstein-deSitter universe) which is $a \propto t^{2/3}$.
- The solution to the equation for $\Omega_0 > 1$ (closed) is

$$a = \frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos\Theta),$$

$$t = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}} (\Theta - \sin\Theta).$$

- The solution for $\Omega_0 < 1$ (open) is

$$a = \frac{\Omega_0}{2(1 - \Omega_0)} (\cosh\Theta - 1),$$

$$t = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{3/2}} (\sinh\Theta - \Theta).$$

- For a closed universe, when $\Theta = 2\pi$, we have $H_0 t = \pi\Omega_0/(\Omega_0 - 1)^{3/2}$, $a = 0$, i.e., the universe recollapses to singularity. In the case of an open universe, a increases indefinitely.

Matter-dominated static universe

- Einstein initially believed that the universe was static. He tried to obtain the solution using the metric

$$ds^2 = dt^2 - R^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

  where R now is a constant. The Einstein tensor for this metric is clearly

$$G^0_{\ 0} = 3\frac{k}{R^2}, \quad G^1_{\ 1} = G^2_{\ 2} = G^3_{\ 3} = \frac{k}{R^2}.$$

- Now if we assume the universe to be filled with matter, then $T^0_{\ 0} = \rho$, $T^1_{\ 1} = T^2_{\ 2} = T^3_{\ 3} = -P = 0$ and hence the Einstein equation becomes

$$3\frac{k}{R^2} = 8\pi G\rho, \qquad \frac{k}{R^2} = 0.$$

- Clearly no sensible solution exists for this spacetime, and hence there is *no* static homogeneous isotropic model filled with non-relativistic matter.

Cosmological constant

- To find the solution, Einstein realised that one can always add a constant to the Einstein tensor and still satisfy the Bianchi identities (that was the motivation for the form of G_{ik} in the first place).
- Hence he modified his equations to

$$R_{ik} - \frac{1}{2} g_{ik} R - \Lambda g_{ik} = 8\pi G T_{ik},$$

  where Λ is a constant, known as the **cosmological constant**.
- Then we get the solutions in the form

$$3\frac{k}{R^2} - \Lambda = 8\pi G\rho, \qquad \frac{k}{R^2} - \Lambda = 0.$$

- The second equation gives $\Lambda = k/R^2$, which when put in the first, we get $2k/R^2 = 8\pi G\rho$. Since $\rho > 0$, we must have $k = +1$, i.e., a closed universe. This also implies that one requires $\Lambda > 0$.
- The solution for the scale factor can be written as

$$R = \sqrt{\frac{k}{\Lambda}} = \sqrt{\frac{k}{4\pi G\rho}}.$$

- The static model of Einstein was abandoned after Hubble's observations. Hence the requirement for Λ went away.

Equation of state of Λ

- We can write the modified Einstein equation as

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G \left(T_{ik} + \frac{\Lambda}{8\pi G} g_{ik} \right).$$

- In this manner Λ is interpreted as a source of gravity. Even when no matter is present $T_{ik} = 0$, we have some contribution to the gravitational energy, which is called the **vacuum energy**.
- The corresponding components of the stress-energy tensor for Λ would be

$$T^i{}_j = \frac{\Lambda}{8\pi G} \, \delta^i{}_j = \text{diag} \left(\frac{\Lambda}{8\pi G}, \frac{\Lambda}{8\pi G}, \frac{\Lambda}{8\pi G}, \frac{\Lambda}{8\pi G} \right)$$

giving

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad P_\Lambda = -\frac{\Lambda}{8\pi G}.$$

- So this source has a equation of state $w_\Lambda = -1$. The corresponding density ρ_Λ does not evolve with time.

Matter and cosmological constant

- However, present day observations show that the universe is flat and contains a matter component $\Omega_{m,0}$ and a cosmological constant $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$. In that case

$$H^2(a) = H_0^2 \left[\frac{\Omega_{m,0}}{a^3} + (1 - \Omega_{m,0})\right] \Longrightarrow \dot{a}^2 = \frac{H_0^2}{a} \left[\Omega_{m,0} + (1 - \Omega_{m,0}) a^3\right].$$

- So the solution is

$$H_0 t = \int \frac{\mathrm{d}a \sqrt{a}}{\sqrt{\Omega_{m,0} + (1 - \Omega_{m,0}) a^3}}. = \frac{1}{\sqrt{\Omega_{m,0}}} \int \frac{\mathrm{d}a \sqrt{a}}{\sqrt{1 + K a^3}},$$

  where $K = (1 - \Omega_{m,0})/\Omega_{m,0}$.
- The solution is

$$a(t) = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3} \left[\sinh\left(\frac{3}{2}\sqrt{1 - \Omega_{m,0}}\, H_0 t\right)\right]^{2/3}.$$

- When $t \to 0$, we get $a \propto t^{2/3}$, while at late times $a \propto \mathrm{e}^{\sqrt{1 - \Omega_{m,0}}\, H_0 t}$.
- The acceleration is

$$\ddot{a} = -\frac{4\pi G}{3}\, a \left(\rho_{m,0} a^{-3} + \rho_\Lambda + 3 P_\Lambda\right) = -\frac{H_0^2}{2} a \left(\Omega_{m,0} a^{-3} - 2\Omega_{\Lambda,0}\right).$$

- Thus the universe decelerates for small a and starts accelerating for

$$a > \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3} = \left[\frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})}\right]^{1/3}.$$

Non-flat models with cosmological constant

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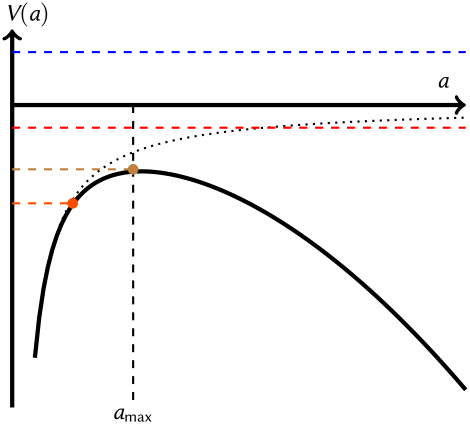
- Consider a universe with matter $\Omega_{m,0}$ and a cosmological constant $\Omega_{\Lambda,0}$, but we allow for non-flat models so that $\Omega_{k,0} = 1 - \Omega_{m,0} - \Omega_{\Lambda,0}$.
- In that case,
$$H^2(a) = H_0^2 \left(\frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{\Omega_{k,0}}{a^2} \right) \implies \frac{1}{2} \frac{\dot{a}^2}{H_0^2} + \left(-\frac{\Omega_{m,0}}{2a} - \frac{\Omega_{\Lambda,0} a^2}{2} \right) = \frac{\Omega_{k,0}}{2}.$$
- This resembles the motion in a potential
$$V(a) = -\frac{\Omega_{m,0}}{2a} - \frac{\Omega_{\Lambda,0} a^2}{2}$$
with constant energy $E = \Omega_{k,0}/2$. The time coordinate is scaled to $H_0 t$.
- The potential is always negative with $V(a) \to -\infty$ for $a \to 0, \infty$. It has a maximum at
$$a = a_{\rm max} = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} \right)^{1/3} \implies V_{\rm max} = -\frac{3}{2^{5/3}} \, \Omega_{m,0}^{2/3} \, \Omega_{\Lambda,0}^{1/3}.$$
- The energy E is positive for open models $\Omega_{k,0} > 0$, and negative for closed models $\Omega_{k,0} < 0$.

Dynamics of the non-flat models with Λ



- Let us consider models which start at $a = 0$ at $t = 0$.
- When $E > 0$ (open models), the scale factor will expand to ∞.
- When $E < 0$ (closed models), but $E > V_{\rm max}$, the scale factor will still expand to ∞.
- In these expanding models, the universe decelerates for $a < a_{\rm max}$ and accelerates afterwards.
- On the other hand, when $E < V_{\rm max}$, the scale factor reaches a maximum and then recollapses to 0.
- When $E = V_{\rm max}$, the universe expands and approaches a static state at $a = a_{\rm max}$. This is the *Einstein's static universe.*