Cosmology
Lecture 4

FLRW kinematics: geodesics

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Geodesic equation

- Let us consider the motion of particles moving along geodesics in the FLRW metric.
- Because of the presence of factors like $S_k(\chi)$, the standard Lagrangian method becomes quite complicated in this case. There are easier ways to do this problem.
- Let us start with the conventional geodesic equation (λ being the affine parameter)

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = 0,$$

- Using the derivative $u^i = \mathrm{d}x^i/\mathrm{d}\lambda$

$$\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} + \Gamma^i_{jk} u^j u^k = 0 \Longrightarrow u^i_{;k} u^k = 0.$$

- In terms of the covariant derivative of $u_i = g_{ij} u^j$, we have

$$u_{i;k} u^k = (g_{ij} u^j)_{;k} u^k = g_{ij} u^j_{;k} u^k = 0.$$

- Let us expand this to obtain a different form of the geodesic equation

$$\begin{aligned} 0 &= \left(\frac{\partial u_i}{\partial x^k} - \Gamma^j_{ik} u_j\right) u^k = \frac{\partial u_i}{\partial x^k} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \Gamma^j_{ik} \left(g_{jl} u^l\right) u^k \\ &= \frac{\mathrm{d}u_i}{\mathrm{d}\lambda} - g_{jl} \Gamma^j_{ik} u^l u^k = \frac{\mathrm{d}u_i}{\mathrm{d}\lambda} - \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right) u^l u^k \\ &= \frac{\mathrm{d}u_i}{\mathrm{d}\lambda} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} u^l u^k. \end{aligned}$$

Conserved quantities

- So, the geodesic equation to be used has the form

$$\frac{\mathrm{d}u_i}{\mathrm{d}\lambda} = \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i} u^l u^k,$$

which shows that if the metric is independent of a particular x^i, then u_i will be conserved along the geodesic.

- Now, let the geodesic pass through some point P. Since the space is homogeneous and isotropic, we can take P to be the origin $r = \chi = 0$ without any loss of generality.
- Consider $x^i = \phi$. The metric is independent of ϕ, so $\mathrm{d}u_\phi/\mathrm{d}\lambda = 0$ and hence u_ϕ is constant $u_\phi = C_3$.
- But

$$u_\phi = g_{\phi\phi} u^\phi = -R^2(t) S_k^2(\chi) \sin^2\theta \; u^\phi.$$

- Now since the geodesic passes through P where $\chi = 0$, we have $u_\phi = 0$ at P. Since it is a constant, $u_\phi = C_3 = 0$ along the geodesic, or, $u^\phi = g^{\phi\phi} u_\phi = 0$ or, $\phi = \text{const}$.
- For $x^i = \theta$, we have

$$\frac{\mathrm{d}u_\theta}{\mathrm{d}\lambda} = \frac{1}{2}\frac{\partial g_{kl}}{\partial \theta} u^l u^k = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial \theta} u^\phi u^\phi = 0.$$

- Hence u_θ is constant and is given by

$$u_\theta = g_{\theta\theta} u^\theta = -R^2(t) S_k^2(\chi) u^\theta = 0,$$

and thus $\theta = \text{const}$.

The radial and time equation

- For $x^i = \chi$, we have
$$\frac{\mathrm{d}u_\chi}{\mathrm{d}\lambda} = \frac{1}{2}\frac{\partial g_{kl}}{\partial \chi} u^l u^k = \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial \chi} u^t u^t + \frac{\partial g_{\chi\chi}}{\partial \chi} u^\chi u^\chi\right) = 0.$$
- Hence we will have
$$u_\chi = g_{\chi\chi} u^\chi = -R^2(t)\,\frac{\mathrm{d}\chi}{\mathrm{d}\lambda} = \mathrm{const} = K.$$
- If the initial conditions are such that $K = 0$, then we have $\chi = \mathrm{const}$, which is the case of fundamental observers. For other geodesics, we only have $R^2\ \mathrm{d}\chi/\mathrm{d}\lambda = \mathrm{const}$.
- The final equation is obtained from the constraint $c^2(\mathrm{d}t/\mathrm{d}\lambda)^2 - R^2(t)(\mathrm{d}\chi/\mathrm{d}\lambda)^2 = 0$ or 1, depending on whether the particle is massless or massive.
- For photons we have
$$c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = \frac{K^2}{R^2(t)},$$
- For massive particles we get (using the proper time s as the affine parameter λ)
$$c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 1 + \frac{K^2}{R^2(t)}.$$

The three-momentum and energy of massive particles

- The three-momentum of a massive particle is

$$p^\alpha = m \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = m u^\alpha .$$

- The magnitude of the three-momentum is then given by

$$|\vec{p}|^2 = -g_{\alpha\beta} p^\alpha p^\beta = -m^2 u^\alpha u_\alpha = -m^2 g^{\chi\chi} u_\chi u_\chi = \frac{m^2 K^2}{R^2(t)} .$$

- Hence $|\vec{p}| \propto 1/R(t)$ for massive particles.
- If the massive particle is non-relativistic $E \ll mc^2$, then its (kinetic) energy $E_{\mathrm{NR}} \propto \vec{p}^2 \propto R^{-2}(t)$.
- On the other hand, if it moves with ultra-relativistic speeds $E \gg mc^2$, then $E \propto |\vec{p}| \propto R^{-1}(t)$.

The three-momentum and energy of photons

- For photons, the energy is given by
$$E^2 = \left(u^0\right)^2 = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = \frac{K^2}{c^2 R^2(t)}.$$
- Thus $E \propto R^{-1}(t)$, which also implies that $\nu \propto R^{-1}(t)$. This is the redshift of light.
- The photon momentum is
$$|\vec{p}|^2 = -\frac{1}{c^4} g_{\chi\chi} u^\chi u^\chi = -\frac{1}{c^4} g^{\chi\chi} u_\chi u_\chi = \frac{K^2}{c^4 R^2(t)}.$$
  Thus, $E^2 = c^2 |\vec{p}|^2$, as expected.
- Also $|\vec{p}| \propto R^{-1}(t)$, similar to massive particles.

Volume element

- The spatial (proper) volume element, i.e., the proper volume of the region of space lying in the infinitesmial coordinate range $(r, r+\mathrm{d}r)$, or $(\chi, \chi+\mathrm{d}\chi)$, and subtending an infinitesimal solid angle $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ at the observer, is

$$\mathrm{d}V_P = \frac{R^3(t) r^2 \sin\theta}{\sqrt{1-kr^2}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = R^3(t) S_k^2(\chi)\sin\theta\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

- The volume of the spherical shell of thickness dr is obtained by integrating over the solid angle

$$\mathrm{d}V_P = \frac{4\pi R^3(t) r^2}{\sqrt{1-kr^2}}\,\mathrm{d}r = 4\pi R^3(t) S_k^2(\chi)\,\mathrm{d}\chi.$$

- The *comoving volume element* is given by the same quantity evaluated at $t = t_0$

$$\mathrm{d}V = \frac{4\pi R_0^3 r^2}{\sqrt{1-kr^2}}\,\mathrm{d}r = 4\pi R_0^3 S_k^2(\chi)\,\mathrm{d}\chi.$$

- The importance of the comoving volume is this: imagine a set of objects that are fundamental observers, i.e., their χ (or r) coordinates are fixed. If these objects are neither created or destroyed, then the number of objects per comoving volume will remain constant.
- The number per proper volume will decrease as $R^{-3}(t)$ since the volume increases with time.

Source counts

- Let $n(t)$ be the number of galaxies per unit *comoving volume* at some epoch t.
- Then the number of galaxies out to a distance $\chi_{\rm max}$ is given by

$$N(\chi_{\rm max}) = \int_{\chi=0}^{\chi_{\rm max}} \mathrm{d}V \, n(t) = 4\pi R_0^3 \int_0^{\chi_{\rm max}} \mathrm{d}\chi \; n(t) S_k^2(\chi),$$

  where the relation between t and χ is given by $\chi = c \int_t^{t_0} \mathrm{d}t'/R(t')$.
- If galaxies are neither created nor destroyed, then $n(t) = n_0$ is a constant.
- If each galaxy has a luminosity L, then the flux received from each is $L/(4\pi d_L^2)$. The total flux received is

$$F_{\rm max} = 4\pi R_0^3 n_0 \int_0^{\chi_{\rm max}} \mathrm{d}\chi \; S_k^2(\chi) \frac{L}{4\pi R_0^2 S_k^2(\chi)(1+z)^2} = L R_0 n_0 \int_0^{\chi_{\rm max}} \frac{\mathrm{d}\chi}{(1+z)^2}.$$

- Use $\mathrm{d}\chi = -c\mathrm{d}t/R(t)$ and $1 + z = R_0/R(t)$ to write

$$F_{\rm max} = -c \; L n_0 \int_{t_0}^{t_{\rm min}} \mathrm{d}t \frac{R(t)}{R_0} = c \; L n_0 \int_{t_{\rm min}}^{t_0} \mathrm{d}t \frac{R(t)}{R_0}.$$

- If we integrate over the whole age of the universe, we get

$$F_{\rm max} = c \; L n_0 \int_0^{t_0} \mathrm{d}t \frac{R(t)}{R_0}.$$

Olber's paradox

- Now, suppose we assume $R(t)/R_0 = (t/t_0)^\alpha$, we get

$$F_{\max} = ct_0 \ Ln_0 \frac{1}{\alpha + 1},$$

which is clearly finite for an expanding universe $\alpha > 0$.

- This is very significant result as it resolves what is known as the **Olber's paradox**.
- The paradox states that for a *static Euclidean universe*, the night-sky should be blazing with light. If we assume that the number density of galaxies n_0 to be independent of time, and if every galaxy radiates with a luminosity L, then the resulting flux from galaxies within a radius $r_{\max}$ is

$$F_{\max} = \int_0^{r_{\max}} \mathrm{d}r \, 4\pi r^2 \ n_0 \ \frac{L}{4\pi r^2} = Ln_0 \, r_{\max}.$$

- Clearly, this blows up for an infinite universe $r_{\max} \to \infty$.
- Alternatively, the Olber's paradox is stated as that in a static universe, every line of sight must end up on a luminous star.
- It turns out that the darkness of night-sky is a consequence of finite age of the universe and expansion (which causes redshift).

Particle horizon

- Now, let us consider a comoving observer O situated (without loss of generality) at $\chi = 0$.
- Suppose that a second comoving observer E has coordinate χ_E and emits a photon at cosmic time t_E, which reaches O at time t. The comoving coordinate χ_E of the emitter E is determined by

$$\chi_E = c \int_{t_E}^{t} \frac{\mathrm{d}t'}{R(t')}.$$

- Assuming light to be the fastest possible signal, the only signals emitted at time t_E that O receives by the time t are from radial coordinates $\chi < \chi_E$.
- If we take $t_E \to 0$, the corresponding χ_E sets the limit of points which could have come in causal contact with O at time t. This limit of the vision of the universe is known as the **particle horizon**.
- At any given cosmic time t, the χ-coordinate of the particle horizon is given by

$$\chi_{\text{hor}}(t) = c \int_{0}^{t} \frac{\mathrm{d}t'}{R(t')}.$$

- The corresponding proper distance to the particle horizon is $d_{\text{hor}}(t) = R(t)\chi_{\text{hor}}(t)$. Note that $d_{\text{hor}}(t) \neq c\, t$.
- For a universe with $R(t) = R_0(t/t_0)^\alpha$, we get

$$d_{\text{hor}}(t) = \frac{ct}{1-\alpha}, \quad \text{when } \alpha < 1,$$

and $d_{\text{hor}}(t) \to \infty$ when $\alpha \geq 1$. In general, the horizon is larger when expansion is faster.

Blackbody radiation

- For a blackbody, the number of photons per unit volume within a frequency range $[\nu, \nu + \mathrm{d}\nu]$ is given by

$$n_\nu \mathrm{d}\nu = \frac{8\pi\nu^2}{\mathrm{e}^{h_P\nu/k_B T} - 1}\mathrm{d}\nu.$$

- Suppose at the epoch labeled by expansion parameter $R_1 \equiv R(t_1)$, the universe contains a uniform sea of blackbody radiation at temperature T_1.
- The number of photons in a three-dimensional *proper volume* $\mathrm{d}V_1$ in the frequency range $[\nu_1, \nu_1 + \mathrm{d}\nu_1]$ is $\mathrm{d}N = n_{\nu_1}(R_1)\mathrm{d}\nu_1 \mathrm{d}V_1$.
- At the later epoch R_2, each of these photons has been redshifted to frequency $\nu_2 = \nu_1 R_1/R_2$.
- Also, the volume under consideration has expanded to $\mathrm{d}V_2 = \mathrm{d}V_1 R_2^3/R_1^3$.
- Assuming the photons to be conserved, we must have $\mathrm{d}N = n_{\nu_2}(R_2)\mathrm{d}\nu_2 \mathrm{d}V_2$, i.e.,

$$n_{\nu_1}(R_1)\mathrm{d}\nu_1 \mathrm{d}V_1 = n_{\nu_2}(R_2)\mathrm{d}\nu_2 \mathrm{d}V_2 = n_{\nu_2}(R_2)\frac{R_1}{R_2}\mathrm{d}\nu_1 \frac{R_2^3}{R_1^3}\mathrm{d}V_1 = n_{\nu_2}(R_2)\frac{R_2^2}{R_1^2}\mathrm{d}\nu_1 \mathrm{d}V_1,$$

- This leads to

$$n_{\nu_2}(R_2) = n_{\nu_1}(R_1)\frac{R_1^2}{R_2^2} = \frac{R_1^2}{R_2^2}\frac{8\pi\nu_1^2}{\mathrm{e}^{h_P\nu_1/k_B T_1} - 1} = \frac{R_1^2}{R_2^2}\frac{R_2^2}{R_1^2}\frac{8\pi\nu_2^2}{\mathrm{e}^{h_P\nu_2 R_2/k_B T_1 R_1} - 1} = \frac{8\pi\nu_2^2}{\mathrm{e}^{h_P\nu_2/k_B T_2} - 1}.$$

  Thus n_{ν_2} also is a blackbody function, with the redshifted temperature $T_2 = T_1 \ R_1/R_2$.
- Hence, the temperature of a blackbody evolves as $T \propto R^{-1}(t)$. This is indeed the case for CMB.

Radiation energy density

- The distribution function, which is the energy per unit volume within a frequency range $[\nu, \nu + \mathrm{d}\nu]$, is given by

$$u_\nu \mathrm{d}\nu = \frac{8\pi h_P \nu^3}{\mathrm{e}^{h_P\nu/k_B T} - 1} \mathrm{d}\nu.$$

  This is also the specific intensity of a blackbody.
- The total blackbody radiation per unit volume (energy density) is

$$\rho c^2 = \int_0^\infty \mathrm{d}\nu \, \frac{8\pi h_P \nu^3}{\mathrm{e}^{h_P\nu/k_B T} - 1} = \frac{8\pi k_B^4}{h_P^3} \frac{\pi^4}{15} T^4 \propto T^4$$

- Since $T \propto R^{-1}(t)$, we have $\rho \propto R^{-4}(t)$.
- Note the difference with normal matter where the density scales as $R^{-3}(t)$.
- In the case of radiation, the number density scales as $R^{-3}(t)$ as expected, however, there is an additional factor of $R^{-1}(t)$ because of the redshift of energies.

Cosmological fluids

- We idealize the universe as filled with a perfect fluid which, at large scales, must be homogeneous.
- The fluid must be at rest in the preferred cosmological frame, for otherwise its velocity would allow us to distinguish one spatial direction from another and the universe would not be isotropic.
- In general, the stress-energy tensor of the fluid will be given by

$$T^i{}_j = (\rho c^2 + P) u^i u_j - P\, \delta^i{}_j.$$

- In the cosmological rest frame, the fluid four-velocity is $u^i = (1, 0, 0, 0)$, and the stress-energy tensor will take the form

$$T^i{}_j = \mathrm{diag}(\rho c^2, -P, -P, -P).$$

- Because of homogeneity, all fluid properties depend only on time, $\rho = \rho(t)$, $P = P(t)$ etc.

The conservation equation

- Let us consider the equation of motion for matter (or the conservation equation) $T^i{}_{j;i} = 0$.
- Because of isotropy, the spatial components of this equation must vanish identically.
- For the $j = 0$ component, we have

$$
\begin{aligned}
0 = T^0{}_{0;0} + T^\alpha{}_{0;\alpha} &= \frac{\partial T^0{}_0}{\partial x^0} + \Gamma^0_{k0} T^k{}_0 - \Gamma^k_{00} T^0{}_k + \frac{\partial T^\alpha{}_0}{\partial x^\alpha} + \Gamma^\alpha_{k\alpha} T^k{}_0 - \Gamma^k_{\alpha 0} T^\alpha{}_k \\
&= \frac{\partial T^0_0}{\partial x^0} + \Gamma^0_{00} T^0{}_0 - \Gamma^0_{00} T^0{}_0 + \Gamma^\alpha_{0\alpha} T^0{}_0 - \Gamma^\beta_{\alpha 0} T^\alpha_\beta = \frac{\mathrm{d}(\rho c^2)}{\mathrm{d}t} + 3\frac{\dot{R}}{R}(\rho c^2 + P).
\end{aligned}
$$

- To understand the physical significance of the conservation equation, note that

$$
\frac{\mathrm{d}(\rho c^2 R^3)}{\mathrm{d}t} = R^3 \left(\frac{\mathrm{d}(\rho c^2)}{\mathrm{d}t} + 3\frac{\dot{R}}{R}\rho c^2 \right) = -3R^2 \dot{R} P = -P \frac{\mathrm{d}(R^3)}{\mathrm{d}t}.
$$

- This has the form

$$
\mathrm{d}E + P \mathrm{d}V = 0,
$$

which is the first law of thermodynamics (conservation of energy).

The evolution of energy density

- Let us assume that the pressure is related to the density by

$$P = w\,\rho c^2.$$

  This is often known as the **equation of state**.
- Then the conservation equation gives

$$0 = \frac{\mathrm{d}(\rho c^2)}{\mathrm{d}t} + 3\frac{\dot{R}}{R}(\rho c^2 + P) = c^2\dot{\rho} + 3c^2(1+w)\frac{\dot{R}}{R}\rho \Longrightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R}.$$

- The solution is $\ln\rho = -3(1+w)\ln R +$ const (for $w \neq -1$) or,

$$\rho \propto R^{-3(1+w)}.$$

  For $w = -1$, we have $\rho =$ const.

Different types of cosmological fluids: dust and radiation

- **Dust:** This represents a set of particles that are rest with respect to each other. Hence they are at rest in the cosmological frame, i.e., they are a collection of fundamental observers.
- These have no random motions and hence $P = 0$. This gives $w = 0$ and hence $\rho \propto R^{-3}$ (consistent with the expansion of the volume).
- **Radiation:** The other extreme is radiation characterized by the specific intensity I_ν.
- The energy density is
$$\rho_\nu c^2 = \frac{1}{c} \int \mathrm{d}\Omega \; I_\nu,$$
while the pressure is
$$P_\nu = \frac{1}{c} \int \mathrm{d}\Omega \; \cos^2\theta \; I_\nu.$$
- For an isotropic radiation, we get
$$P_\nu = \frac{I_\nu}{c} \times 2\pi \times \frac{2}{3} = \frac{4\pi}{3} \frac{I_\nu}{c} = \frac{1}{3} \rho_\nu c^2.$$
- Thus, for the cosmological radiation fluid, we have $w = 1/3$ and hence $\rho \propto R^{-4}$. This agrees with what was derived for the blackbody radiation.

Different types of cosmological fluids: ideal gas

- **Non-relativistic ideal gas:** Consider an ideal gas consisting of non-relativistic particles of mass m.
- Then the rest mass density and pressure are

$$\rho_m = \frac{mN}{V}, \quad P = \frac{Nk_BT}{V} = \frac{k_BT}{m}\rho_m,$$

where N is the number of particles in a volume V.
- Note that $w \neq k_BT/mc^2$ because $\rho \neq \rho_m$. There is a kinetic energy term which needs to be accounted for.
- The kinetic energy is nothing but the internal energy. For a gas with adiabatic index γ, it is given by

$$\frac{N}{V}\frac{k_BT}{\gamma - 1}.$$

This leads to

$$P = (\gamma - 1)(\rho - \rho_m)c^2.$$

- Then

$$\rho c^2 = \rho_m c^2 + \frac{P}{\gamma - 1} = \frac{mc^2}{k_BT}P + \frac{P}{\gamma - 1} = P\,\frac{mc^2}{k_BT}\left[1 + \frac{1}{\gamma - 1}\frac{k_BT}{mc^2}\right],$$

which gives the equation of state to be

$$w = \frac{k_BT}{mc^2}\left[1 + \frac{1}{\gamma - 1}\frac{k_BT}{mc^2}\right]^{-1}.$$

- For a non-relativistic gas $k_BT \sim m\langle v^2\rangle \ll mc^2$, hence $w \ll 1$. In this case too, $\rho \propto R^{-3}$.