Cosmology
Lecture 3

FLRW kinematics: redshift and distances

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Physical and comoving distances

- Since we will be talking about observations, let us write the metric putting back the quantity c

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - R^2(t) \left[\mathrm{d}\chi^2 + S_k^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)\right]$$
$$= c^2 \mathrm{d}t^2 - R^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)\right]$$

- The **physical or proper distance** to a point with coordinate r is obtained by putting $\mathrm{d}t = \mathrm{d}\theta = \mathrm{d}\phi = 0$

$$d_P = R(t)\chi = R(t) \int \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = R(t) S_k^{-1}(r).$$

- The **comoving distance** to the same point is defined as the distance if it was *measured at the present epoch* and is given by

$$d_C = R_0 \chi = R_0 S_k^{-1}(r).$$

- Clearly, the proper distance between two fundamental observers increases $\propto R(t)$, while the comoving distance remains constant:

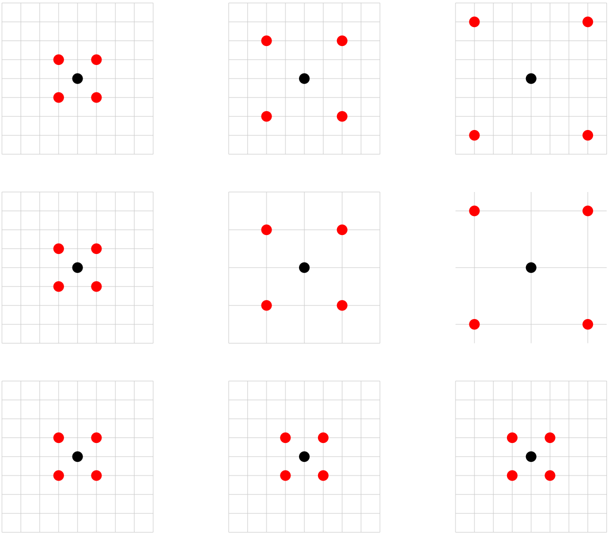
$$d_P = \frac{R(t)}{R_0} d_C.$$

The coordinate systems

Physical/proper coordinates

Comoving coordinates

Comoving coordinates

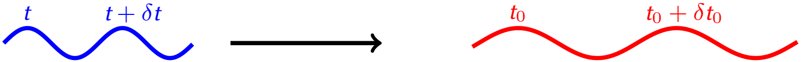


Emission and receiving of electromagnetic wave

- The propagation of photons (radially) is governed by the equation

$$0 = \mathrm{d}s^2 = c^2\mathrm{d}t^2 - R^2(t)\mathrm{d}\chi^2 \implies \mathrm{d}\chi/\mathrm{d}t = -c/R(t)$$

  where the negative sign implies "incoming" photons.
- Consider a wavecrest which is emitted at t from some distant galaxy situated at coordinates χ. This signal is received by an observer on earth at the present epoch t_0.



- The next wavecrest is emitted at $t + \delta t$ and is received at $t_0 + \delta t_0$.
- The comoving distance travelled by light between the two points is just the comoving distance to the galaxy and is given by

$$R_0 \chi = R_0 c \int_t^{t_0} \frac{\mathrm{d}t'}{R(t')} = R_0 c \int_{t+\delta t}^{t_0+\delta t_0} \frac{\mathrm{d}t'}{R(t')},$$

Cosmological time dilation

- The integral can be broken into three parts using

$$\int_t^{t_0} = \int_t^{t+\delta t} + \int_{t+\delta t}^{t_0+\delta t_0} - \int_{t_0}^{t_0+\delta t_0} \Longrightarrow \int_t^{t+\delta t} \frac{\mathrm{d}t'}{R(t')} = \int_{t_0}^{t_0+\delta t_0} \frac{\mathrm{d}t'}{R(t')}.$$

Now, if R does not change over the time-scales of δt and δt_0, we can take it out of the integral and hence

$$\frac{\delta t}{R(t)} = \frac{\delta t_0}{R_0}.$$

- We have assumed that $R(t)$ does not change significantly over the interval(s) δt, i.e., $\dot{R}/R\,\delta t \ll 1$ (this implies age of the Universe $\sim R/\dot{R} \gg \delta t$, the time-period of the wave).
- Since $R_0 > R(t)$, we have $\delta t_0 > \delta t$.
- This is simply the **cosmological time dilation**. Events observed take longer (“stretched”) than they happen in their rest frame.

Cosmological redshift

- We have $\delta t / R(t) = \delta t_0 / R_0$.
- Now, the frequency of the light wave is simply $\nu = 1/\delta t$. We thus obtain

$$\frac{\nu_0}{\nu} = \frac{R(t)}{R_0} \implies \frac{\lambda_0}{\lambda} = \frac{R_0}{R(t)}.$$

- The **redshift** is defined as

$$z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{\lambda_0}{\lambda} - 1.$$

  Thus the redshift is related to the scale factors by the relation

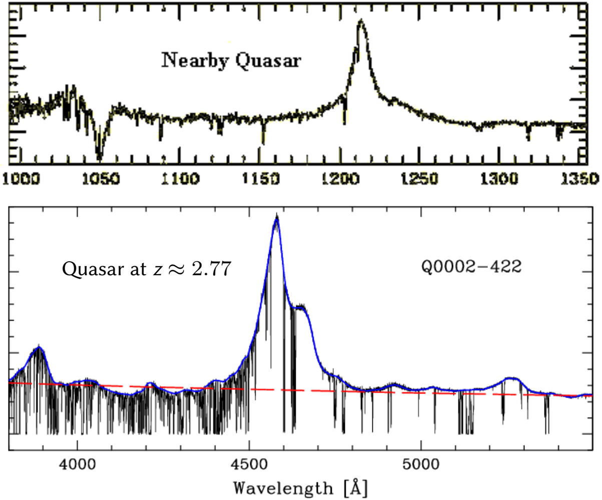
$$1 + z = \frac{R_0}{R(t)}.$$

- This implies that if we can measure the redshift of a light signal originating from a distant galaxy, we can estimate the size of the Universe (relative to today) when the signal originated.
- Measurement of z, along with the knowledge of the function $R(t)/R_0$, allows us to estimate t when the light was emitted.
- Similarly, knowledge of t and $R(t)$ allows us to calculate the distance

$$d_P = R(t)\chi = c\, R(t) \int_t^{t_0} \frac{\mathrm{d}t'}{R(t')}.$$

- Often z is used as a proxy for time and also distance. Present epoch corresponds to $z = 0$.

Example of redshifts: quasars (Lyman-α emission line)



Note that according to this interpretation, the redshift is simply a consequence of expansion of the spacetime.

Hubble-Lemaitre law

- If we assume that a fundamental observer (galaxy) is at a coordinate distance χ, its proper distance is

$$d_P(t) = R(t)\chi.$$

- The velocity with which it is moving away is

$$v_P = \dot{R}(t)\chi = H(t)d_P, \quad H(t) \equiv \dot{R}/R.$$

  $H(t)$ is the **Hubble function/parameter**.
- If the galaxy is close to us, then the time of measurement corresponds to $t \approx t_0$ and hence we recover Hubble's law in its traditional form $v_P = H_0 d_P$.
- Note that $[H] = 1/t$. Hence $H^{-1}(t)$ defines a time-scale.
- The significance of this time-scale can be understood if we assume that the Universe expands as a power-law

$$R(t) = R_0 \left(\frac{t}{t_0}\right)^{\alpha} \implies H(t) = \frac{\alpha}{t}, \;\; H_0 = \frac{\alpha}{t_0}.$$

- Hence $H(t)$ approximately measures the age of the Universe at the epoch t. Its present value is written as

$$H_0 = 100\ h \text{ km s}^{-1} \text{ Mpc}^{-1},$$

  with $h \approx 0.7$ (measured). The corresponding time-scale is

$$t_0 \approx 10^{10} h^{-1} \text{yrs},$$

  which is roughly the age of the universe.

Comoving distance in terms of z

- Since z is directly observable, it is convenient if all quantities are expressed as functions of z.
- Let us first express $R(t)$ in terms of z. This is easy as we have

$$R(t) = \frac{R_0}{1+z}.$$

- Next, we need to express χ in terms of z. Since $\mathrm{d}\chi/\mathrm{d}t = -c/R(t)$ for photons coming towards us, we have

$$\chi = \int_0^{\chi} \mathrm{d}\chi' = -c \int_{t_0}^{t} \frac{\mathrm{d}t'}{R(t')}$$

- We already know to express $R(t)$ in terms of z. We only need to express $\mathrm{d}t$ in terms of $\mathrm{d}z$. We can do this as

$$\mathrm{d}z = \mathrm{d}(1+z) = \mathrm{d}\left(\frac{R_0}{R}\right) = -\frac{R_0}{R^2}\,\mathrm{d}R = -\frac{R_0}{R^2}\,\dot{R}\,\mathrm{d}t = -\frac{R_0}{R}\frac{\dot{R}}{R}\,\mathrm{d}t = -(1+z)\,H(z)\,\mathrm{d}t.$$

- Hence the comoving distance is

$$d_C = R_0\chi = -c\,R_0 \int_{t_0}^{t} \frac{\mathrm{d}t'}{R(t')} = +c\,R_0 \int_0^{z} \frac{\mathrm{d}z'}{(1+z')H(z')} \times \frac{1+z'}{R_0} = c \int_0^{z} \frac{\mathrm{d}z'}{H(z')}.$$

- Often, $c/H(z)$ is called the **Hubble distance**, then the comoving distance is just the integral of the Hubble distance.

Proper distance in terms of z

- The proper distance is related to the redshift through the relation

$$d_P(z) = \frac{R(t)}{R_0} d_C(z) = \frac{c}{1+z} \int_0^z \frac{\mathrm{d}z'}{H(z')}.$$

  Clearly, this is not the simple Hubble-Lemaitre law.
- In fact, Hubble derived his law of expanding universe as $z = H_0 d_P / c$ but his observations were limited to galaxies with redshifts $z < 0.003$.
- When $z \ll 1$, we can assume that $H(z)$ is almost constant and is equal to its present value H_0:

$$d_P(z) \approx \frac{c}{H_0} \int_0^z \mathrm{d}z' = \frac{c\, z}{H_0}.$$

Acceleration of the expansion

- To understand how the Hubble-Lemaitre law is modified for slightly higher values of z, let us expand in a power series and retain the next order terms.
- Let us start with the expansion around $t = t_0$

$$
\begin{aligned}
R(t) &\approx R_0 + (t - t_0)\dot{R}_0 + \frac{1}{2}(t - t_0)^2 \ddot{R}_0 + \ldots \\
&= R_0 + (t - t_0) \left.\frac{\dot{R}}{R}\right|_{t_0} R_0 + \frac{1}{2}(t - t_0)^2 \left.\frac{\ddot{R}\,R}{\dot{R}^2}\right|_{t_0} \left.\frac{\dot{R}^2}{R^2}\right|_{t_0} R_0 + \ldots \\
&= R_0 \left[1 + (t - t_0)H_0 - \frac{1}{2}(t - t_0)^2 q_0 H_0^2 + \ldots\right],
\end{aligned}
$$

  where $q_0 = -\ddot{R}_0\ R_0/\dot{R}_0^2$.
- Note that the acceleration of the expansion is measured by the quantity $\ddot{R}$. It is customary to define the **deceleration parameter** as

$$
q(t) \equiv -\frac{\ddot{R}\,R}{\dot{R}^2} = -\frac{\ddot{R}}{R}\frac{1}{H^2}.
$$

- Also note that the derivative of $H(t)$ can be expressed in terms of q as

$$
\dot{H}(t) = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} = -q(t)H^2(t) - H^2(t) = -H^2(t)[1 + q(t)].
$$

Series expansions in z

- Often it is useful to make expansions in powers of z.
- The derivation of the series expansion of $d_P(z)$ is obtained from the following sequence:
  1. Using the series of $R(t)$, obtain the expansion for z:
  $$z(t) = \frac{R_0}{R(t)} - 1 = H_0(t_0 - t) + (t_0 - t)^2 H_0^2 \left(1 + \frac{q_0}{2}\right) + \ldots .$$
  2. Invert it to obtain
  $$t_0 - t = H_0^{-1} \left[z - \left(1 + \frac{q_0}{2}\right) z^2 + \ldots\right].$$
  3. Finally, expand $1/H$ in terms of t and then use the above expansion to get
  $$\frac{1}{H(z)} = \frac{1}{H_0} - \frac{\dot{H}_0}{H_0^2}(t - t_0) + \ldots = \frac{1}{H_0} - (1 + q_0) H_0^{-1} z + \ldots .$$
- Putting this in the expression for $d_P(z)$, we obtain the result
$$d_P(z) = \frac{c}{1+z} \int_0^z \frac{\mathrm{d}z'}{H(z')} = \frac{c}{H_0} \left[z - \frac{1}{2}(3 + q_0) z^2 + \ldots\right].$$
- The lowest order term is the Hubble law. However, there are higher order corrections for larger values of z which depend on the derivatives of H.
- The comoving distance as a series expansion in z is
$$d_C(z) = c \int_0^z \frac{\mathrm{d}z}{H(z)} = d_P(z)(1+z) = \frac{c}{H_0} \left[z - \frac{1}{2}(1 + q_0) z^2 + \ldots\right].$$

Look-back time and age

- The look-back time is given by

$$t_0 - t = \int_t^{t_0} \mathrm{d}t = \int_0^z \frac{\mathrm{d}z}{(1+z)H(z)}.$$

- The age is given by

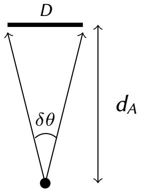
$$t = \int_0^t \mathrm{d}t = \int_z^\infty \frac{\mathrm{d}z}{(1+z)H(z)}.$$

Angular diameter distance

- Unfortunately, there is no direct way of measuring the proper or comoving distance to an object.
- In cosmology, the distance to an object far away can be measured via observations in more than one ways.
- The first one is to measure the angular size of the object, and if we somehow know its intrinsic size (say it is a "standard ruler"), we can estimate its distance. This is known as the **angular diameter distance**.
- Assuming the object has a proper size D and subtends an angle $\delta\theta$, then its distance in Euclidean geometry would be $d_A = D/\delta\theta$. This is the operational definition of the angular diameter distance.
- The proper transverse size D of a object subtending an angle $\delta\theta$ at distance χ is obtained by putting $\mathrm{d}t = \mathrm{d}r = \mathrm{d}\phi = 0$:

$$D = R(t) S_k(\chi) \delta\theta,$$

  where t is the time at which the photon was emitted from χ.



- The angular diameter distance is thus

$$d_A(t) \equiv \frac{D}{\delta\theta} = R(t) S_k(\chi).$$

- In terms of z, this becomes

$$d_A(z) = \frac{R_0 S_k(\chi)}{1+z}, \quad \chi = \frac{c}{R_0} \int_0^z \frac{\mathrm{d}z'}{H(z')}.$$

- Note that for flat universe ($k = 0$)

$$d_A(z) = \frac{c}{1+z} \int_0^z \frac{\mathrm{d}z'}{H(z')} = d_P(z)$$

  is independent of R_0.

Luminosity distance

- A second way of defining distance would be to use the flux-luminosity relation.
- In Euclidean geometry, the luminosity L (of an isotropic source) and the observed flux F are related by

$$F = \frac{L}{4\pi d_L^2}.$$

This is the operational definition of the **luminosity distance** d_L.

- For simplicity, let us assume the emitter is monochromatic.
- The luminosity is the energy emitted per unit time

$$L \equiv \frac{\delta E}{\delta t} = \frac{\delta N_\gamma \; h_P \nu}{\delta t},$$

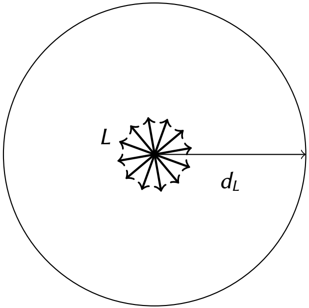
where δN_γ is number of photons emitted.

- The flux is defined as the energy received per unit time per unit area

$$F \equiv \frac{\delta N_\gamma \; h_P \nu_0}{\delta A \, \delta t_0},$$

where we have assumed that frequency and the time interval may change because of expansion.

- In the Euclidean case, $\nu_0 = \nu$, $\delta t_0 = \delta t$ and $\delta A = 4\pi d_L^2$, hence we recover the familiar relation.



Luminosity distance (contd)

- Now, there are three effects which have to be accounted for
  1. The photons emitted from a source at time t at distance χ, while reaching us, would be distributed over a sphere of surface area
  $$\delta A = 4\pi R_0^2 r^2 = 4\pi R_0^2 S_k^2(\chi).$$
  2. The frequency of the photons would be shifted to $\nu \to \nu_0 = \nu\, R(t)/R_0 = \nu/(1+z)$.
  3. The arrival time interval would be changed to $\delta t_0 = \delta t\, R_0/R(t) = \delta t\,(1+z)$.
- So we have
$$F = \frac{\delta N_\gamma\ h_P \nu_0}{\delta A\, \delta t_0} = \frac{\delta N_\gamma\ [h_P \nu/(1+z)]}{4\pi R_0^2 S_k^2(\chi)\ [\delta t\,(1+z)]} = \frac{L}{4\pi R_0^2 S_k^2(\chi)(1+z)^2}.$$
- This implies that the luminosity distance will be given by
$$d_L(z) = R_0 S_k(\chi)(1+z).$$
- Note that in general $d_L(t) \neq d_A(t) \neq d_P(t) \neq d_C(t)$. In fact $d_L(z) = d_A(z)\ (1+z)^2$.
- In modern days, the Hubble-Lemaitre law is represented in terms of $d_L(z)$. Let us first expand
$$S_k(\chi) = \frac{\sin\left(\sqrt{k}\,\chi\right)}{\sqrt{k}} = \chi - \frac{k}{6}\chi^3 + \ldots = \frac{c\, H_0^{-1}}{R_0}\left[z - \frac{1}{2}(1+q_0)z^2\right] - \mathcal{O}(z^3) + \ldots$$
- Then
$$d_L(z) = \frac{c}{H_0}\left[z + \frac{1}{2}\,(1-q_0)\,z^2 + \ldots\right].$$

Distance modulus

- In optical, UV, NIR bands, luminosities and fluxes are measured using the **magnitude system**.
- The **apparent magnitude** of an object is defined in terms of the observed flux

$$m = -2.5 \log_{10}(F/F_0)$$

  where F_0 is a constant chosen based on some pre-determined convention.
- For example, one can choose Vega to represent magnitude zero so that $F_0 = F_{\rm vega}$. In recent times, other conventions are used too (e.g., AB-magnitude).
- Similarly, the **absolute magnitude** is defined in terms of the luminosity by a similar relation

$$M = -2.5 \log_{10}(L/L_1).$$

- Clearly,

$$M = -2.5 \log_{10}\left(4\pi d_L^2 F/L_1\right) = -2.5 \log_{10}\left(F/F_0\right) - 2.5 \log_{10}\left(4\pi d_L^2 F_0/L_1\right) = m - 2.5 \log_{10}\left(4\pi d_L^2 F_0/L_1\right)$$

- The constant is chosen such that the absolute magnitude equals the apparent magnitude the object would have if it were at a standard distance (10 parsec) away from the observer. Hence $L_1 = 4\pi(10\text{pc})^2 F_0$ and

$$M = m - 5 \log_{10}\left(d_L/10\text{pc}\right).$$

- A related quantity is

$$m - M = 5 \log_{10}\left(d_L/10\text{pc}\right)$$

  which is known as the **distance modulus**. It is a measure of the luminosity distance to the source.

K-correction

- In general, we observe only in a limited frequency range $[\nu_1, \nu_2]$. In Euclidean space, the bandpass flux is

$$F_{\rm BP} = \frac{1}{4\pi d_L^2} \int_{\nu_1}^{\nu_2} {\rm d}\nu \; L_\nu(\nu).$$

- We can define $m_{\rm BP} = -2.5 \log_{10}(F_{\rm BP}/F_{0,\rm BP})$ and $M_{\rm BP} = -2.5 \log_{10} \left[\int_{\nu_1}^{\nu_2} {\rm d}\nu \; L_\nu(\nu)/L_{1,\rm BP}\right]$, with $L_{1,\rm BP} = 4\pi(10{\rm pc})^2 F_{0,\rm BP}$ to obtain the standard distance modulus relation $m_{\rm BP} - M_{\rm BP} = 5 \log_{10}(d_L/10{\rm pc})$.
- In an expanding universe, redshift implies that the detected light was actually emitted at higher frequencies

$$F_{\rm BP} = \frac{1}{4\pi d_L^2} \int_{\nu_1(1+z)}^{\nu_2(1+z)} {\rm d}\nu \; L_\nu(\nu).$$

- Assuming the same relations for $m_{\rm BP}$ and $M_{\rm BP}$ as in the Euclidean case, we can show that

$$m_{\rm BP} - M_{\rm BP} = 5 \log_{10}(d_L/10{\rm pc}) + K(z),$$

  where the extra correction, known as **K-correction**, is

$$K(z) = -2.5 \log_{10} \left[\frac{\int_{\nu_1(1+z)}^{\nu_2(1+z)} {\rm d}\nu \; L_\nu(\nu)}{\int_{\nu_1}^{\nu_2} {\rm d}\nu \; L_\nu(\nu)}\right] = -2.5 \log_{10}(1+z) - 2.5 \log_{10} \left[\frac{\int_{\nu_1}^{\nu_2} {\rm d}\nu \; L_\nu[\nu(1+z)]}{\int_{\nu_1}^{\nu_2} {\rm d}\nu \; L_\nu(\nu)}\right].$$

- This correction is important while comparing properties of galaxies at different redshifts.
- For a source with $L_\nu \propto \nu^{-\alpha}$, we can show that $K(z) = 2.5(\alpha - 1) \log_{10}(1+z)$. Thus sources with $\alpha \approx 1$ (say, quasars) have negligible correction.