Cosmology: Assignment 3

IUCAA-NCRA Graduate School
January - February 2018

01 February 2018
To be returned in the class on 13 February 2018

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Relativistic perturbations:** According to the relativistic theory of linear perturbations, the ${}^0_0$ component of the Einstein equations (in Fourier space) is

$$3\frac{a'}{a}\phi' + \left(k^2 + 3\frac{a'^2}{a^2}\right)\phi = -4\pi G a^2 \bar{\rho}\delta,$$

while the ${}^\alpha_\alpha$ component is

$$\phi'' + 3\frac{a'}{a}\phi' + 2\frac{a''}{a}\phi - \frac{a'^2}{a^2}\phi = 4\pi G a^2 p.$$

Here $'$s denote derivatives with respect to the conformal time η, and ϕ is the gravitational potential. Other quantities have their usual meaning as discussed in the class. We have assumed the metric to be spatially flat.

Assume there exists only one component of matter given by the equation of state $P = w\rho$ ($w \neq -1$). The equation holds for the unperturbed as well as the perturbed quantities.

(a) Use the solutions to the background universe to find the quantities a'/a and a''/a in terms of η.

(b) Show that ϕ satisfies the second order differential equation

$$\phi'' + \frac{6(1+w)}{(1+3w)\eta}\phi' + wk^2\phi = 0.$$

(c) Find the solution $\phi(\eta)$ to the above equation for large scales $k\eta \to 0$.

[4+5+3]

2. **Newtonian perturbations:** The equations for a non-relativistic fluid are given by

$$\begin{aligned}
\text{Continuity equation:} \quad & \dot{\rho}(t,\boldsymbol{r}) + \boldsymbol{\nabla}_r \cdot [\rho(t,\boldsymbol{r})\ \boldsymbol{U}(t,\boldsymbol{r})] = 0, \\
\text{Euler equation:} \quad & \dot{\boldsymbol{U}}(t,\boldsymbol{r}) + [\boldsymbol{U}(t,\boldsymbol{r}) \cdot \boldsymbol{\nabla}_r]\boldsymbol{U}(t,\boldsymbol{r}) = -\boldsymbol{\nabla}_r \Phi(t,\boldsymbol{r}) - \frac{\boldsymbol{\nabla}_r P(t,\boldsymbol{r})}{\rho(t,\boldsymbol{r})}, \\
\text{Poisson equation:} \quad & \boldsymbol{\nabla}_r^2 \Phi(t,\boldsymbol{r}) = 4\pi G \rho(t,\boldsymbol{r}),
\end{aligned}$$

where the overdot represents partial derivative $\partial/\partial t$ with respect to the time t and $\boldsymbol{\nabla}_r$ is the spatial gradient operator with respect to the proper coordinates $\boldsymbol{r}$. The fluid density and pressure are denoted by $\rho(t,\boldsymbol{r})$ and $P(t,\boldsymbol{r})$, respectively, while the proper velocity of the fluid is $\boldsymbol{U}(t,\boldsymbol{r}) \equiv \mathrm{d}\boldsymbol{r}/\mathrm{d}t$. The quantity $\Phi(t,\boldsymbol{r})$ is the gravitational potential.

One can write the equations in terms of the comoving coordinate $\boldsymbol{x}$

$$\boldsymbol{r} = a(t)\boldsymbol{x},$$

and the perturbed quantities

$$\begin{aligned}
\text{Density contrast} \qquad & \delta(t,\boldsymbol{x}) = \frac{\rho(t,\boldsymbol{x})}{\bar{\rho}(t)} - 1, \\
\text{Peculiar velocity field} \qquad & \boldsymbol{v}(t,\boldsymbol{x}) \equiv a(t)\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{U}(t,\boldsymbol{x}) - \frac{\dot{a}}{a}\boldsymbol{r}, \\
\text{Perturbed pressure} \qquad & p(t,\boldsymbol{x}) = P(t,\boldsymbol{x}) - \bar{P}(t), \\
\text{Perturbed gravitational field} \qquad & \phi(t,\boldsymbol{x}) = \Phi(t,\boldsymbol{x}) - \bar{\Phi}(t,\boldsymbol{x}).
\end{aligned}$$

The symbols with bars denote the average values of the corresponding quantities, which are independent of the spatial coordinates except $\bar{\Phi}$, which satisfies the equation for the smooth universe

$$\boldsymbol{\nabla}_r^2\bar{\Phi} = 4\pi G\bar{\rho}$$

(a) Use the above to first separate out the "unperturbed" part of the equations and write them down.

(b) Show the perturbed parts of the fluid equations are given by

$$\begin{aligned}
\dot{\delta} + \frac{1}{a}\,\boldsymbol{\nabla}\cdot[(1+\delta)\,\boldsymbol{v}] &= 0, \\
\dot{\boldsymbol{v}} + \frac{\dot{a}}{a}\,\boldsymbol{v} + \frac{1}{a}\,(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} &= -\frac{1}{a}\,\boldsymbol{\nabla}\phi - \frac{\boldsymbol{\nabla}p}{a\bar{\rho}(1+\delta)}, \\
\boldsymbol{\nabla}^2\phi &= 4\pi G\bar{\rho}a^2\delta,
\end{aligned}$$

where we are using the convention

$$\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_x.$$

(c) Show that for a pressureless fluid ($p = 0$), the density contrast, under the *linear approximation*, satisfies the equation

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\bar{\rho}\delta.$$

[5+8+5]