Cosmology: Assignment 0
IUCAA-NCRA Graduate School
January – February 2018

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- The questions in this assignment are based on standard topics you would have covered till now.
- You may look up textbooks and/or consult friends for solving the problems, but make sure you understand the solutions.
- You need *not* submit this assignment. However, if you find any of these questions nontrivial/difficult, please let me know so that the rest of the course can be designed appropriately.

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1. **General Theory of Relativity:** Calculate the Christoffel symbols, the components of the Ricci tensor R_{ik} and the Einstein tensor G_{ik} for the metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right].$$

2. **Thermodynamics:** Starting with the second law of thermodynamics

$$T\ \mathrm{d}S = \mathrm{d}E + P\mathrm{d}V - \mu\mathrm{d}N,$$

show that

$$S = \frac{E + PV - \mu N}{T}.$$

3. **Statistical Mechanics:** Consider the (relativistic) phase space distribution for some species A:

$$f_A(p)\ \mathrm{d}^3p = \frac{g_A}{(2\pi)^3}\frac{\mathrm{d}^3p}{\mathrm{e}^{[E(p)-\mu_A]/T_A} \pm 1},$$

where

$$E(p) = \sqrt{p^2 + m_A^2}.$$

The quantity g_A is the spin-degenracy factor for the species, μ_A is the chemical potential and T_A is the temperature. The upper sign corresponds to fermions and the lower one to bosons. For simplicity, we use units where $\hbar = c = k_B = 1$.

Derive expressions for the number density n_A, energy density ρ_A, pressure P_A, and entropy density $s_A = (\rho_A + P_A - n_A\mu_A)/T_A$ in the ultra-relativistic limit $T_A \gg m_A$, $E_A \gg m_A$ and in the non-relativistic limit $T_A \ll m_A$.

4. **Fluids:** The evolution of the phase space distribution $f(\boldsymbol{r}, \boldsymbol{p}, t)$ of a collection of microscopic particles is given by the Boltzmann equation

$$\frac{\mathrm{d}f(\boldsymbol{r},\boldsymbol{p},t)}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{r}} f\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\nabla}_{\boldsymbol{p}} f\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = C[f]$$

where $C[f]$ denotes the change in the distribution function arising from collisions between the particles. Take moments of this equation and derive the continuity and Euler equations for fluids.

5. **Statistics:** The two point correlation function of a density field is defined as

$$\xi(\boldsymbol{x} - \boldsymbol{x}') \equiv \langle\delta(\boldsymbol{x})\ \delta(\boldsymbol{x}')\rangle,$$

where $\delta(\boldsymbol{x})$ is the density contrast and $\langle\cdots\rangle$ denotes the ensemble average. The Fourier transform of $\delta(\boldsymbol{x})$ is defined as

$$\delta(\boldsymbol{k}) = \int \mathrm{d}^3x\ \delta(\boldsymbol{x})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.$$

Show that

$$\left\langle\delta(\boldsymbol{k})\ \delta^*(\boldsymbol{k}')\right\rangle = (2\pi)^3\ \delta_D(\boldsymbol{k} - \boldsymbol{k}')\ \int \mathrm{d}^3x\ \xi(\boldsymbol{x})\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.$$

6. **Radiation:** Consider the radiative transfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\kappa_\nu \ I_\nu + j_\nu.$$

Show that the formal solution of the above can be written as

$$I_\nu(\tau_\nu) = I_\nu(0) \ \mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} \mathrm{d}\tau_\nu' \ S_\nu(\tau_\nu') \ \mathrm{e}^{(\tau_\nu' - \tau_\nu)},$$

where

$$\tau_\nu = \int_{s_0}^{s} \mathrm{d}s' \ \kappa_\nu(s'),$$

and

$$S_\nu = \frac{j_\nu}{\kappa_\nu}.$$

Assume $I_\nu = I_\nu(0)$ at $s = s_0$.

7. **Classical Mechanics:** Consider the evolution of a spherical shell of radius R which encloses a mass M. The equation of motion is

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM}{R^2}.$$

Show that the first integral of motion is given by

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = E,$$

where E is the integration constant.

Solve the above equation and plot the function $R(t)$ for different values of E. You can choose the initial condition to be $R \to 0$ as $t \to 0$.