Extra-Galactic Astronomy I (Cosmology)
Project 3: Relativistic linear perturbation theory
Total marks: 30

1 Background

The large scale structures we observe in the present day Universe (e.g., through galaxy redshift surveys) originated from small perturbations in the cosmic density and velocity fields present at very early epochs. The initial evolution of the perturbations, when their amplitudes were much smaller than unity, can be described by the linear perturbation theory. In this project, you will be expected to solve the differential equations for the relativistic perturbations in non-relativistic matter and radiation at different length scales, and study their evolution as a function of time.

2 What to do

2.1 Setting up the equations

Assuming the Universe to be consisting of only radiation (r), dark matter (m) and a cosmological constant (Λ), the linear perturbation equations in the Fourier space for a wave vector $\boldsymbol{k}$ can be written as

$$\begin{aligned} \dot{\delta}_m &= k^2 V_m - 3\dot{\Phi}, \\ \dot{V}_m &= -\frac{\dot{a}}{a} V_m + \Phi, \\ \dot{\delta}_r &= \frac{4}{3}(k^2 V_r - 3\dot{\Phi}), \\ \dot{V}_r &= -\frac{1}{4}\delta_r + \Phi, \end{aligned} \tag{1}$$

where the overdots represent derivatives with respect to the conformal time η and we have taken $c = 1$. The cosmological constant has been assumed to not exhibit any inhomogeneities. The quantity δ represents the density contrast, while V represents the measure of the velocity field (the actual velocity field is $\boldsymbol{v} = -\mathrm{i}\boldsymbol{k}V$). The gravitational potential is given by Φ and it satisfies the relativistic Poisson equation

$$3\frac{\dot{a}}{a}\dot{\Phi} + \left(k^2 - 3\kappa + 3\frac{\dot{a}^2}{a^2}\right)\Phi = 4\pi G a^2(\bar{\rho}_m \delta_m + \bar{\rho}_r \delta_r). \tag{2}$$

Remember that the Friedmann equation is given by

$$\frac{\dot{a}^2}{a^2} + \kappa = \frac{8\pi}{3} G a^2 \left(\bar{\rho}_m + \bar{\rho}_r + \bar{\rho}_\Lambda\right). \tag{3}$$

Now change the independent variable from η to

$$y = \frac{a}{a_{\rm eq}}, \tag{4}$$

where $a_{\rm eq}$ is the scale factor at the matter-radiation equality. Write the differential equations in terms of y.

2.2 Solving the differential equations

To set up the initial conditions at $y \to 0$, we assume $\Phi(0)$ to be independent of k. Also, assuming the perturbations to be adiabatic, we write the other initial conditions as

$$\begin{aligned} \delta_r(0) &= 2\Phi(0), \\ \delta_m(0) &= \frac{3}{4}\delta_r(0) = \frac{3}{2}\Phi(0), \\ V_m(0) = V_r(0) &= 0. \end{aligned} \tag{5}$$

Set up a code for solving the set of differential equations numerically for the above initial conditions for three sets of cosmological parameters, i.e., (i) $\Omega_m = 1, \Omega_\Lambda = 0$, (ii) $\Omega_m = 0.3, \Omega_\Lambda = 0$, (iii) $\Omega_m = 0.3, \Omega_\Lambda = 0.7$. You should take Ω_r that is consistent with the CMBR with present day temperature 2.73 K.

2.3 Results to be produced

You need to produce the following plots for the three sets of cosmological parameters given above:

1. Plot $\Phi(k, a)/\Phi(0)$ vs a for three values of $k = 0.001, 0.045, 2h/\text{Mpc}$. The range of a should be between 10^{-8} and 1.
2. Plot $\delta_m(k, a)/\delta_m(k, 1)$ vs a for three values of $k = 0.001, 0.045, 2h/\text{Mpc}$. The range of a should be between 10^{-8} and 1.
3. Plot the transfer function
$$T(k, a) = \frac{\Phi(k, a)}{\Phi(k, 0)} \times \frac{\Phi(k \to 0, 0)}{\Phi(k \to 0, a)} \tag{6}$$
vs k for three values of $a = 10^{-8}, 10^{-4}, 1$. Compare your result for $a = 1$ with the transfer function fit given by BBKS (search the literature to find what this means).