Cosmology: Exercise Sheet 3

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1. At the microscopic level, a fluid can be thought of as a collection of 'molecules'. Ignoring the internal structure of the molecules, we can specify the state of any molecule of mass m by giving its position $\boldsymbol{r}$ and momentum $\boldsymbol{k} = m\boldsymbol{u}$. Let $\mathrm{d}n = f(\boldsymbol{r}, \boldsymbol{k}, t)\mathrm{d}^3x \ \mathrm{d}^3k$ denote the number of molecules in a phase volume $\mathrm{d}^3x \ \mathrm{d}^3k$ at time t. The evolution of this distribution function is given by:

$$\frac{\mathrm{d}f(\boldsymbol{r}, \boldsymbol{k}, t)}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{r}} f \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = C[f],$$

where $C[f]$ denote the change in the distribution function arising from collisions between molecules and can be expressed in terms of the scattering cross section for molecular collisions.

(i) Show that the above equation can be written, in the component form, as

$$\frac{\partial f}{\partial t} + u^a \frac{\partial f}{\partial r^a} - m \frac{\partial \phi}{\partial r^a} \frac{\partial f}{\partial k^a} = C[f].$$

(ii) Multiply by m and integrate over d^3k and show that the equation reduces to the standard equation of continuity.

(iii) Similarly, multiply the equation by k^b and integrate over d^3k to obtain the Euler equation.

2. For the spherical collapse model, compute the form of $\delta(t)$ as $t \to 0$.

3. Usually it is assumed that the virialization of a spherically collapsing object occurs at the epoch when $R \to 0$ in the equations. This corresponds to the parameter θ having a value 2π. Instead, one can assume that the virialization occurs at the epoch when the virialization condition is satisfied, i.e., $\dot{R}^2/2 = GM/(2R)$.

(i) Show that this corresponds to $\theta = 3\pi/2$.

(ii) Calculate the actual density contrast and the linearly extrapolated density contrast for this case.

4. Let the ensemble average of the density contrast field be $\langle \delta(\boldsymbol{x}) \rangle$ and the volume average be

$$\delta_X(\boldsymbol{x}) = \frac{1}{V} \int_V \mathrm{d}^3x' \ \delta(\boldsymbol{x} + \boldsymbol{x'}),$$

where V is the volume centered at $\boldsymbol{x}$ with $X \propto V^{1/3}$ being the linear size of the volume.

(i) If $\langle \delta(\boldsymbol{x}) \rangle = 0$, then show that $\langle \delta_X(\boldsymbol{x}) \rangle = 0$.

(ii) Write the explicit form of the volume average when V is a sphere of radius X.

(iii) Show that

$$\langle \delta_X^2(\boldsymbol{x}) \rangle = \left(\frac{4\pi X^3}{3} \right)^{-1} \int_0^X \mathrm{d}x' \ 4\pi x'^2 \ \xi(2x') \ \ 8 \left(1 - \frac{x'}{X}\right)^2 \left(1 + \frac{x'}{2X}\right),$$

where $\xi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) = \langle \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2) \rangle$ is the correlation function.

(iv) Define a scale of decorrelation X_d such that

$$\xi(2y) = \langle \delta(\boldsymbol{x} - \boldsymbol{y})\delta(\boldsymbol{x} + \boldsymbol{y}) \rangle = 0 \qquad \text{when } y > X_d,$$

i.e., the field becomes decorrelated at scales larger than X_d. Then show that for $X > X_d$,

$$\left\langle \delta_X^2(\boldsymbol{x}) \right\rangle = \frac{X_d^3}{X^3} \langle \delta_{X_d}^2 \rangle.$$

Interpret the result.