Cosmology: Exercise Sheet 2
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1. Consider a galaxy at a proper distance of $d_P(t_0) = 10^9$ light years ≈ 300 Mpc away from us at the present epoch. Since the age of the universe is $t_0 \approx 1.5 \times 10^{10}$ years, there has been sufficient time to exchange about 15 light signals with the galaxy. At earlier times, when the scale factor was smaller, everything was closer together and so we might have naively expected that this would improve causal contact.

   (i) At the epoch of recombination $t = t_{\rm rec}$ (when the cosmic microwave background photons were emitted) the redshift z was approximately 1000. What was the proper distance to the 'galaxy' is $d_P(t_{\rm rec})$?

   (ii) If we assume, for simplicity, that after $t_{\rm rec}$ the expansion followed a matter-dominated universe, and that prior to $t_{\rm rec}$ the expansion followed a radiation-dominated model, what is the proper distance to the causal horizon? How does it compare with $d_P(t_{\rm rec})$ calculated in part (i)?

2. Consider the Lagrangian of a scalar field Φ

$$L = \frac{1}{2} g^{ik} \frac{\partial \Phi}{\partial x^i} \frac{\partial \Phi}{\partial x^k} - V(\Phi),$$

   where $V(\Phi)$ is the potential under which the field evolves. Assume that Φ is a function of only the time coordinate. Using the definition of the stress-energy tensor, show that the density and the pressure associated with the scalar field is given by

$$P_\Phi = \frac{1}{2}\dot{\Phi}^2 - V(\Phi), \quad \rho_\Phi = \frac{1}{2}\dot{\Phi}^2 + V(\Phi).$$

   Under what conditions does this field behave like a cosmological constant?

3. (i) Show that the collisionless Boltzmann equation for an expanding homogeneous and isotropic universe is given by

$$\frac{\partial f}{\partial t} - H(t)\, p \frac{\partial f}{\partial p} = 0.$$

   (ii) Integrate the above equation over all momenta to obtain the evolution equation for number density n of particles. Hence show that $n \propto a^{-3}$.

   (iii) Similarly, integrate the equation over $\int \mathrm{d}^3p\ E(p)$ to obtain the energy conservation equation.

4. The second law of thermodynamics can be written as

$$T\ \mathrm{d}S(T, \mu, V) = \mathrm{d}E + P\mathrm{d}V - \mu \mathrm{d}N = \mathrm{d}[\rho(T,\mu)V] + P(T,\mu)\mathrm{d}V - \mu \mathrm{d}[n(T,\mu)V].$$

   Assuming $\mathrm{d}S$ to be an exact differential, show that

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{\rho + P}{T} + nT \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mu}{T}\right).$$

   Hence show that, upto an additive constant,

$$S = \frac{(\rho + P - \mu n)V}{T}.$$

5. Starting from the expression for the phase space distribution function, derive expressions for the number density n, the energy density ρ and the pressure P for a collection of non-relativistic particles.

6. Assume a species of particles which have decoupled from the rest and are evolving in an expanding universe in absence of any collisions or interactions. Use Liuoville's theorem to show that their temperature $T \propto a^{-1}$ if they were ultra-relativistic during decoupling. Also show that $T \propto a^{-2}$ in the non-relativistic case.

7. Show that in the epoch where the energy density is dominated by ultra-relativistic particles, the radiation temperature T is related to age t by

$$t = \left(\frac{90}{32\pi^3 G}\right)^{1/2} g_{R,\mathrm{tot}}^{-1/2} \frac{1}{T^2},$$

where we have used units in which $k_B = \hbar = c = 1$. What is the equation in a full dimensional form?

8. (i) Use the Saha equation to write the fraction of free electrons x_e as a function of the radiation temperature T. Plot x_e against T and then against z.

   (ii) Let z_{LLS} be the redshift at which $x_e = 0.1$. Plot z_{LLS} against $\Omega_b h^2$.