Astrophysics: Assignment 7

HRI Graduate School
August - December 2011

29 October 2011
To be returned to the tutor by 10 November 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Luminosity function in terms of magnitude:** Show that the Schechter luminosity function, when written in terms of the absolute magnitude $M = -2.5 \log_{10} L +$ constant, has the form

$$\phi(M) = 0.4 \ \ln 10 \ n_* \ 10^{-0.4(\alpha+1)(M-M_*)} \exp\left[-10^{-0.4(M-M_*)}\right]$$

[3]

2. **de Vaucouleurs law:** Show that, for ellipticals following de Vaucouleurs law, the surface brightness contained within the effective radius r_e is half the total brightness of the galaxy.

[4]

3. **Rotation curve of a disk galaxy:** Consider an infinitely thin circular disk having a surface density $\Sigma(R)$.

(i) Argue that the density of the disk can be written as $\rho(R, z) = \Sigma(R) \ \delta_D(z)$.

(ii) Show that the most general solution of the corresponding Poisson equation can be written as

$$\Phi(R, z) = \int_0^\infty \mathrm{d}k \ S(k) \ \mathrm{e}^{-k|z|} J_0(kR)$$

Write the function $S(k)$ in terms of $\Sigma(R)$.

(iii) If we choose a form of $\Sigma(R)$ appropriate for spiral disk galaxies

$$\Sigma(R) = \Sigma_0 \mathrm{e}^{-R/R_d}$$

show that

$$\Phi(R, z) = -2\pi G \Sigma_0 R_d^2 \int_0^\infty \frac{\mathrm{d}k}{(1 + k^2 R_d^2)^{3/2}} \mathrm{e}^{-k|z|} J_0(kR)$$

(iv) Evaluate the integral for $z = 0$ and show that it is given by

$$\Phi(R, 0) = -\pi G \Sigma_0 R \left[I_0(y) K_1(y) - I_1(y) K_0(y)\right]$$

where

$$y \equiv \frac{R}{2R_d}$$

(v) Evaluate the corresponding circular speed. What are its limiting forms for $R \ll R_d$ and $R \gg R_d$?

$[1 + 7 + 3 + 5 + 3]$

4. **Flat rotation curves:** For a spherical galaxy, determine the density profile $\rho(r)$ for which $v_c(r)$ is independent of r.

[2]

5. **Superluminal motion:** Consider a source of radiation that moves with a speed v at an angle θ to the line of sight.

(i) Show that for a given value of v, the apparent velocity $v_\perp$ is maximum when $\theta = \cos^{-1}(v/c)$, with the maximum value being γv where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

(ii) Show that in the limit $v \to c$, the maximum occurs for $\theta \approx \gamma^{-1}$.

(iii) Show that the necessary condition for superluminal motion to occur (at some angle) is $v > c/\sqrt{2}$.

[3 + 2 + 2]

6. **Relativistic beaming:** Consider an object moving in the x direction with uniform (relativistic) velocity v. Let S and S' be the frames of reference attached with us (the observers) and with the moving object respectively (both assumed inertial). Now the moving object ejects a projectile with velocity u' in its own frame S' making an angle θ' with the x direction. From our frame, it will appear that the projectile is moving with u making an angle θ.

(i) Show that

$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}; \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

(ii) Now, consider the special case in which the projectile is a beam of light emitted by the moving object so that $u' = c$. What is the relation between θ and θ'?

(iii) Evaluate the value of θ for $\theta' = 0, \pi/4, \pi/3, \pi/2$ for three different values of $v/c = 0.1, 0.5, 0.9$. Hence argue that θ will in general be smaller than θ' and the difference between θ and θ' becomes more pronounced as $v \to c$.

(iv) Show that if a relativistically moving object (with $v \sim c$) emits radiation in different directions in its own rest frame, it will appear to us that all the radiation is emitted in the forward direction of its motion within a cone of angle $1/\gamma$.

[4 + 1 + 3 + 2]