Astrophysics: Assignment 2

HRI Graduate School
August - December 2011

25 August 2011
To be returned to the tutor by 10 September 2011

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books. However, you should understand and be clear about every step in the answers. Marks may be reduced if you have not understood what you have written even though the answer is correct.
- Let me or your tutor know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Isotropic radiation field:** Consider a radiation field characterized by specific intensity I_ν. What happens to the net radiation flux when the radiation field is isotropic? How is the pressure related to the energy density when the radiation field is isotropic?

   [1 + 1]

2. **Blackbody radiation:** (i) Show that the energy density per frequency range of blackbody radiation is given by

$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\mathrm{e}^{h\nu/k_BT}-1}$$

   Hence show that the total energy density is given by the **Stefan-Boltzmann law**

$$u = \int \mathrm{d}\nu \; u_\nu = a_B \; T^4$$

   What is the expression for the constant a_B and what is its value in cgs units?

   (ii) Show that the emergent flux from a isotropically emitting blackbody surface is

$$F = \int \mathrm{d}\nu \; F_\nu = \sigma \; T^4$$

   Estimate the value of σ in cgs units. Hint: You may have to use a result obtained later in Problem 3.

   (iii) Show that the entropy S of blackbody radiation is related to the temperature T and volume V by

$$S = \frac{4}{3}a_B T^3 V$$

   (iv) Show that of two blackbody curves, the one with higher temperature lies entirely above the other.

   [2 + 1 + 2 + 1]

3. **Flux from a spherical source:** A spherical source of radiation (with radius R) has a uniform intensity I_ν. Show that the total flux of radiation from the source at a distance r from the centre of the source will be

$$F_\nu = \pi I_\nu \left(\frac{R}{r}\right)^2$$

   What is the flux emergent from the source?

   [3]

4. **Flux from an infinite plane:** An infinite plane radiating surface has a uniform specific intensity I_ν. What is the flux at a point situated at a height h above the surface?

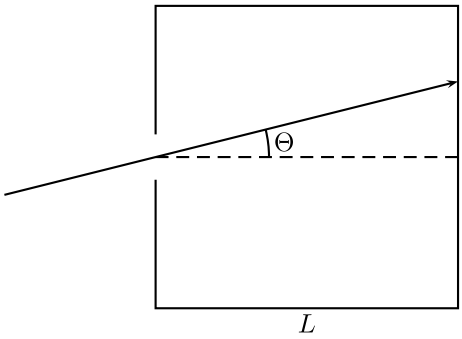
   [3]

5. **Flux in a pinhole camera:** Consider a 'pinhole camera' having a small circular hole of diameter d in its front and having a film plane at a distance L behind it. Show that the flux F_ν at the film plane is related to the incident intensity $I_\nu(\Theta, \Phi)$ in the following way

$$F_\nu = \frac{\pi \cos^4 \Theta}{4 f^2} I_\nu(\Theta, \Phi)$$

where $f = L/d$ is the 'focal ratio.

[4]



6. **Radiative transfer equation in terms of brightness temperature:** Show that the radiative transfer equation for thermal emission (i.e., material with $S_\nu = B_\nu$) in the Rayleigh-Jeans regime has the form

$$\frac{\mathrm{d}T_b}{\mathrm{d}\tau_\nu} = -T_b + T$$

where T is the temperature of the material. Write down the solution when T is constant. What happens when the medium in optically thick?

[3]

7. **Properties of a spherical cloud:** Consider a spherical cloud of gas with a radius R and a constant inside temperature T far away from the observer emitting thermally with a power $P_\nu = \mathrm{d}E/(\mathrm{d}t\ \mathrm{d}\nu\ \mathrm{d}V)$.

(i) Assuming the cloud to be optically thin, find out how the brightness seen by the observer would vary as a function of distance b from the cloud centre.

(ii) What is the flux measured at earth from the entire cloud?

(iii) What is the overall effective temperature of the cloud surface?

(iv) How do the measured brightness temperatures compare with the clouds temperature? You can assume that the frequency of observation $\nu \ll k_B T/h$.

(v) How will the answers to (i)–(iv) be modified if the cloud were optically thick?

$[2 + 2 + 1 + 2 + 3]$

8. **Properties of a supernova remnant:** A supernova remnant has an angular diameter $\Delta\theta = 4.3$ arc minutes and a flux at $\nu = 100$ MHz of $F_{100} = 1.6 \times 10^{-19}$ erg cm^{-2} s^{-1} Hz^{-1}. Assume that the emission is thermal. What is the brightness temperature T_b? What energy regime of the blackbody curve does this correspond to? If the angular size of the source is extremely small, you can take the intensity to be independent of the direction of observation.

[3]

9. **Radiative equilibrium:** (i) Show that the moment of the radiative transfer equation can be written as

$$\frac{\mathrm{d}u_\nu}{\mathrm{d}t} \equiv \frac{\partial u_\nu}{\partial t} + \nabla_{\mathbf{x}} \cdot \mathbf{F}_\nu = 4\pi j_\nu - c\kappa_\nu\ u_\nu$$

which has the form of a "conservation equation" (i.e., change in density + divergence of flux = sources − sinks). Here $\mathbf{F}_\nu$ is the energy flux vector defined as

$$\mathbf{F}_\nu \equiv \int \mathrm{d}\Omega\ I_\nu\ \hat{\mathbf{n}}$$

(ii) The condition of radiative equilibrium is often expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int \mathrm{d}\nu \int \mathrm{d}\Omega\ I_\nu\right) = 0$$

Show that this is equivalent to

$$\int \mathrm{d}\nu \ (4\pi j_\nu - c\kappa_\nu \ u_\nu) = 0$$

which implies that radiation should not add or subtract a net amount of energy from the system. Write the above relation in terms of the source function S_ν. What happens when κ_ν is independent of the frequency ν?

(iii) Hence show that, for static systems ($\partial I_\nu / \partial t \to 0$), the condition for radiative equilibrium can be written as

$$\nabla_{\mathbf{x}} \cdot \mathbf{F} = 0$$

[3 + 1 + 1]

10. **Eddington approximation:** Consider a radiation field which is nearly isotropic. The anisotropy can be introduced by writing the intensity as a power series in μ and retain terms only up to linear

$$I_\nu(\chi, \mu) = a_\nu(\chi) + b_\nu(\chi)\mu$$

Calculate u_ν and P_ν for this field and show that Eddington approximation holds for this nearly isotropic field.

[2]

11. **Solar limb-darkening** The observational data for solar limb-darkening is given below. Compare it with the solution obtained using grey atmosphere and Eddington approximation.

μ	Relative intensity
1.00	1.00
0.90	0.944
0.80	0.898
0.70	0.842
0.60	0.788
0.50	0.750
0.40	0.670
0.30	0.602
0.20	0.522
0.10	0.450

[3]

12. **Hydrogen in ground state:** Calculate the ratio of Hydrogen atoms in the $n = 2$ state with respect to the ground state for 3 different temperatures: $T_1 = 6000$ K, $T_2 = 10^4$ K, $T_3 = 3 \times 10^4$ K. (Assume hydrogen to be neutral although it is likely to be ionized at these temperatures.)

[2]

13. **Derivation of Saha equation:** Consider a system of quantum particles (could be fermions or bosons) called A in equilibrium at a temperature T.

(i) What is the phase space distribution function $f_A(t, \mathbf{x}, \mathbf{p})$?

(ii) Show that the number density of particles is given by

$$n_A(t, \mathbf{x}) \equiv \int \mathrm{d}^3 p \ f_A(t, \mathbf{x}, \mathbf{p}) = \frac{4\pi}{c^3} \int_{m_A c^2}^{\infty} \mathrm{d}E \ E \ \sqrt{E^2 - m_A^2 c^4} \ E \ f_A$$

(iii) Show that, in the non-relativistic limit ($k_B T \ll m_A c^2$), the expression reduces to

$$n_A = \frac{4\pi g_A c^3}{h^3} \ m_A^3 \ \mathrm{e}^{\mu_A / k_B T} \int_1^{\infty} \mathrm{d}y \ \sqrt{y^2 - 1} \ \mathrm{e}^{-y m_A c^2 / k_B T}$$

You may assume $k_B T \ll m_A c^2 - \mu_A$ so that the system is "dilute" (i.e., the occupation numbers are much smaller than unity).

Evaluate the integral using properties of the modified Bessel function of the second kind $K_n(z)$ and show that

$$n_A = \frac{4\pi g_A c k_B}{h^3} \ m_A^2 \ T \ \mathrm{e}^{\mu_A / k_B T} \ K_2\left(\frac{m_A c^2}{k_B T}\right)$$

Then use the asymptotic expression for $K_2(z)$ to show that

$$n_A = g_A \left(\frac{2\pi m_A k_B T}{h^2}\right)^{3/2} \mathrm{e}^{(\mu_A - m_A c^2)/k_B T}$$

(iv) Now consider the equilibrium system $p+e \rightleftharpoons \mathrm{H}+\gamma$. Consider three species $A = e, p, \mathrm{H}$. Write down the expressions for n_e, n_p and n_H.

(v) Finally, show that

$$\frac{n_p\ n_e}{n_H} = \frac{2g_p}{g_H} \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \mathrm{e}^{-B/T}$$

where $B = (m_p + m_e - m_H)c^2$ is the binding energy of hydrogen atom.

[1 + 2 + 5 + 2 + 2]

14. **Effect of Thomson scattering:** Consider an (homogeneous) atmosphere of completely ionized hydrogen having the same density as the density of the Earths atmosphere. Using the fact that a beam of light passing through this atmosphere will be attenuated due to Thomson scattering by free electrons, calculate the (order of magnitude) path length which this beam has to traverse before its intensity is reduced to half its original strength. (This problem should give you an idea of why the cosmological matter-radiation decoupling took place after the number of free electrons was reduced due to the formation of atoms.)

[2]