ASTRONOMY AND ASTROPHYSICS-I: Assignment 2

Department of Physics
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11 September 2019
To be returned in the class on 19 September 2019

- The deadline for the submission of the solutions of this assignment will be strictly enforced. No marks will be given if the assignment is not returned in time.
- You are free to discuss the solutions with friends, seniors and consult any books.
- Let me know if you find anything to be unclear or if you think that something is wrong in any of the questions.

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1. **Isotropic radiation field:** How is the pressure related to the energy density when the radiation field (i.e., the specific intensity) is isotropic?

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2. **Derivation of Saha equation:** Consider a system of quantum particles (could be fermions or bosons) called A in equilibrium at a temperature T.

(a) Write down the phase space distribution function $f_A(t, \vec{x}, \vec{p})$ if A is a fermion. Write down $f_A(t, \vec{x}, \vec{p})$ when A is a boson. You do *not* have to derive the expressions.

(b) Show that the number density of such particles is given by

$$n_A(t, \vec{x}) \equiv \int \mathrm{d}^3p\ f_A(t, \vec{x}, \vec{p}) = \frac{4\pi}{c^3} \int_{m_A c^2}^{\infty} \mathrm{d}E\ E\ \sqrt{E^2 - m_A^2 c^4}\ E\ f_A.$$

(c) Show that, in the non-relativistic limit ($k_B T \ll m_A c^2$), the expression reduces to (both for fermions and bosons)

$$n_A = \frac{4\pi g_A c^3}{h^3}\ m_A^3\ \mathrm{e}^{\mu_A / k_B T} \int_1^{\infty} \mathrm{d}y\ \sqrt{y^2 - 1}\ \mathrm{e}^{-y m_A c^2 / k_B T}.$$

You may assume $k_B T \ll m_A c^2 - \mu_A$ so that the system is "dilute" (i.e., the occupation numbers are much smaller than unity).
The integral in the non-relativistic limit is given by (you do *not* have to show this)

$$n_A = g_A \left(\frac{2\pi m_A k_B T}{h^2}\right)^{3/2} \mathrm{e}^{\cdot}(\mu_A - m_A c^2)/k_B T$$

(d) Now consider the equilibrium system $p + e \rightleftharpoons \mathrm{H} + \gamma$ (photoionization and recombination). Consider three species $A = e, p, \mathrm{H}$. Write down the expressions for n_e, n_p and n_H.

(e) Finally, show that

$$\frac{n_p\ n_e}{n_H} = \frac{2 g_p}{g_H}\ \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2}\ \mathrm{e}^{-B/T},$$

where $B = (m_p + m_e - m_H)c^2$ is the binding energy of hydrogen atom.

[2+4+4+3+3]

3. **Radiative transfer equation in terms of brightness temperature:** Since in the Rayleigh-Jeans regime of the balckbody radiation, the intensity is proportional to the temperature, one often defines a quantity called *brightness temperature* by the relation

$$T_b = \frac{c^2}{2\nu^2 k_B} I_\nu.$$

If the source is a blackbody and the frequency range is in the Rayleigh-Jeans regime, then T_b is independent of ν and is equal to the thermodynamic temperature of the source.

(a) Show that the radiative transfer equation for material with $S_\nu = B_\nu(T)$ in the Rayleigh-Jeans regime has the form

$$\frac{\mathrm{d}T_b(\tau_\nu)}{\mathrm{d}\tau_\nu} = -T_b(\tau_\nu) + T(\tau_\nu),$$

where T is the temperature of the material.

(b) Write down the solution when T is constant.

(c) What happens when the medium in optically thick?

(d) Write the solution to the equation when T is constant and the medium is optically thin.

[2 + 2 + 1 + 2]

4. **Effect of Thomson scattering:** Consider an (homogeneous) atmosphere of completely ionized hydrogen having the same density as the density of the Earth's atmosphere. Using the fact that a beam of light passing through this atmosphere will be attenuated due to Thomson scattering by free electrons, calculate the (order of magnitude) path length which this beam has to traverse before its intensity is reduced to half its original strength.

[4]