Quantum & Statistical Mechanics II

Class Test II : Solutions

Short Questions : 2.5 x 4 = 10.0

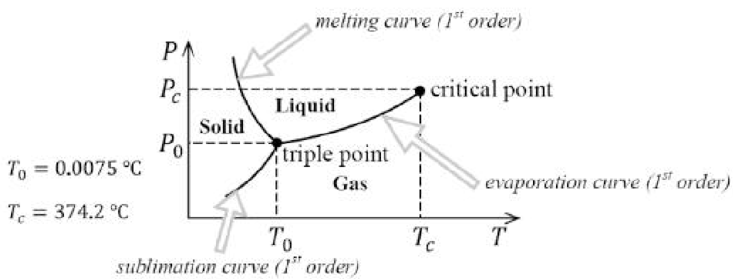
1. A 100-watt heating coil is placed in a vessel containing 1 Kg of water and is switched on. After a while the water attains a steady temperature. If the heating coil is now removed how long would it take for the water to cool by 1°C? (Assume, specific heat of water = 4.5 KJ/Kg.°C)

   Solution : If an amount of material maintains a steady temperature while receiving heat from an external agency, then the amount of heat lost from that material per unit time (in absence of heating) should equal the heat received per unit time from that agency. Now the heat lost per unit time by the material in unit time is given by,

   specific heat x temperature x mass = Energy.

   Therefore, the time required for the water to cool by 1° is : 100 W x t = 4.5 KJ/Kg°C x 1 °C x 1 Kg ⇒ t = 45 s.

2. In a suitable parameter-plane, draw the phase diagram of water, marking all the phases, the triple point and the critical point.



3. For a thermodynamic system of N particles in 3-D, prove Liouville's theorem -

$$\frac{d\rho(p,q)}{dt} = 0\,,$$

   where $\rho(p,q)$ is the phase-space density of a thermodynamic system.

   Solution : For a Hamiltonian system with canonical coordinates q_i and conjugate momenta p_i, where i=1,...,N (each of the indices having D components where D is the dimensionality of the problem) the time derivative of the phase space distribution function $\rho(p,q)$ is zero -

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\rho}{\partial p_i}\dot{p}_i\right) = 0. \tag{1}$$

   This can be proved under the assumption that $\rho(p,q)$ obeys an N-dimensional version of the continuity equation given by,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial(\rho\dot{q}_i)}{\partial q_i} + \frac{\partial(\rho\dot{p}_i)}{\partial p_i}\right) = 0. \tag{2}$$

   Since, the relation

$$\rho\sum_{i=1}^{N}\left(\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i}\right) = \rho\sum_{i=1}^{N}\left(\frac{\partial^2 H}{\partial q_i\,\partial p_i} - \frac{\partial^2 H}{\partial p_i\partial q_i}\right) = 0, \tag{3}$$

   is true for a Hamiltonial system, Liouville's theorem follows.

4. Find the Chandrasekhar mass (appropriate for a White Dwarf) in terms of the relevant fundamental constants.

   Solution : The Chandrasekhar mass is the maximum mass that can be supported (against its own gravity) by the pressure of the degenerate relativistic electrons. Therefore, the issues of physics that are relevant here are - quantum physics, relativity and gravity. Consequently, the fundamental constants that should define this mass are - h, c, G, m_p; where m_p is the mass of the proton (you can also use atomic mass unit).

   Dimensional analysis gives us,

   $$M = D[h^a] \times D[c^b] \times D[G^c] \times D[m_u^d] = (M.L^2.T^{-1})^a \times (L.T^{-1})^b \times (M^{-1}.L^3.T^{-2})^c \times M^d .$$

   It can be seen from here that the combination (hc/G), which has the dimension of mass-square ($\sqrt{\hbar c/G}$ is, in fact, the Planck mass) arises naturally. The Chandrasekhar mass is actually a power of this, weighted by the proton mass, given by

   $$M_{\rm Ch} \simeq \left(\frac{\hbar c}{G m_p^2}\right)^{\frac{3}{2}} m_p.$$

   Full marks if you, at least, obtain the hc/G ratio and comment upon it.

Medium Questions : 5.0 x 3 = 15.0

1. Find the critical point (V_c, T_c) for a van der Waals gas and discuss the behaviour of the system below the critical point.

   Solution : The equation of state for a van der Waals is given by -

   $$P = \frac{NkT}{V-b} - \frac{a}{V^2} .$$

   Therefore, we have -

   $$\frac{dP}{dV} = 0 \Rightarrow \frac{NKT}{(v-b)^2} - \frac{2a}{V^3} = 0 \tag{4}$$

   and,

   $$\frac{d^2P}{dV^2} = 0 \Rightarrow \frac{NKT}{(V-b)^3} - \frac{3a}{V^4} = 0 \tag{5}$$

   Using the above equations we obtain the critical point given by,

   $$V_c = 3b, NkT_c = \frac{8a}{27b} .$$

   When $T > T_c$, isotherms are like standard vapour with $\frac{\partial P}{\partial V} < 0$. However, for $T < T_c$, between two roots of the equation of state, $\frac{\partial P}{\partial V} > 0$ which is unphysical. It implies that the equation of state does not represent a homogeneous substance but must be a mixture of more than one phases. In this situation, two phases co-exist with same P, T, implying a first order phase transition with the existance of a latent heat.

2. Discuss the behaviour of the chemical potential (μ) for a non-interacting gas of - a) fermions and b) bosons. Consider both non-relativistic and ultra-relativistic cases.

   Solution :

   (a) Fermion - μ is equal to the Fermi energy.

   (b) Boson - $\mu \lesssim 0$ for non-relativistic and $\mu < mc^2$ for relativistic particles.

3. Consider a He^2 White Dwarf, with interior density ranging from 10^6 g cm^{-3}on the surface to 10^{10} g cm^{-3}at the centre. Find whether the electrons are relativistic (or not) across this density range. What would be the Fermi temperature of the electrons at the centre?

   Solution : For an N-particle degenerate electron gas the following relation holds -

   $$N = \frac{2}{h^3}\int_V dq \int_{p_F} dp = \frac{2}{h^3} V \frac{4\pi}{3} p_F^3 \Rightarrow n = \frac{8\pi}{3}\left(\frac{p_F}{h}\right)^3 , \tag{6}$$

   where, n is the number density of electrons and the factor of 2 takes care of the spin degree of freedom. Therefore,

   $$p_F = \left(\frac{3n}{8\pi}\right)^{\frac{1}{3}} h = \left(\frac{3\rho}{8\pi m_a \mu}\right)^{\frac{1}{3}} h = 2.19 \times 10^{-19} \rho^{\frac{1}{3}} , \tag{7}$$

where, m_a is the atomic mass unit; μ is the number of massive particles per electron and is 2 for He^2. The electron gas would be relativistic if $p_F/mc \gtrsim 1$. For $\rho \gtrsim 10^6\ \mathrm{g\,cm^{-3}}$, this condition is satisfied and therefore the electrons would be relativistic throughout a He^2 White Dwarf.

The Fermi energy of a degenerate relativistic gas would be given by

$$E_F = \left(m_e^2c^4 + p_F^2c^2\right)^{\frac{1}{2}} = \left(1.66\times 10^{-14} + 4.79\times 10^{-17}\rho^{2/3}\right)^{\frac{1}{2}} = 10^{-7}\left(1.66 + 4.79\times 10^{-3}\rho^{2/3}\right)^{\frac{1}{2}}\ , \tag{8}$$

where m_e is the electron mass. The Fermi temperature at the core is therefore

$$T_F = E_F/k_B = 10^{-7}\left(1.66 + 4.79\times 10^{-3}10^{20/3}\right)^{\frac{1}{2}}/(1.4\times 10^{-16})^\circ C = 1.06\times 10^{11}.^\circ C\,. \tag{9}$$