- A two element interferometer

Astronomical Techniques II : Lecture 5

Ruta Kale

Low Frequency Radio Astronomy (Chp. 4)
http://www.ncra.tifr.res.in/ncra/gmrt/gmrt-users/low-frequency-radio-astronomy

Synthesis imaging in radio astronomy II, Chp 2

Interferometry and synthesis in radio astronomy (Chp 2)

Tools of radio astronomy

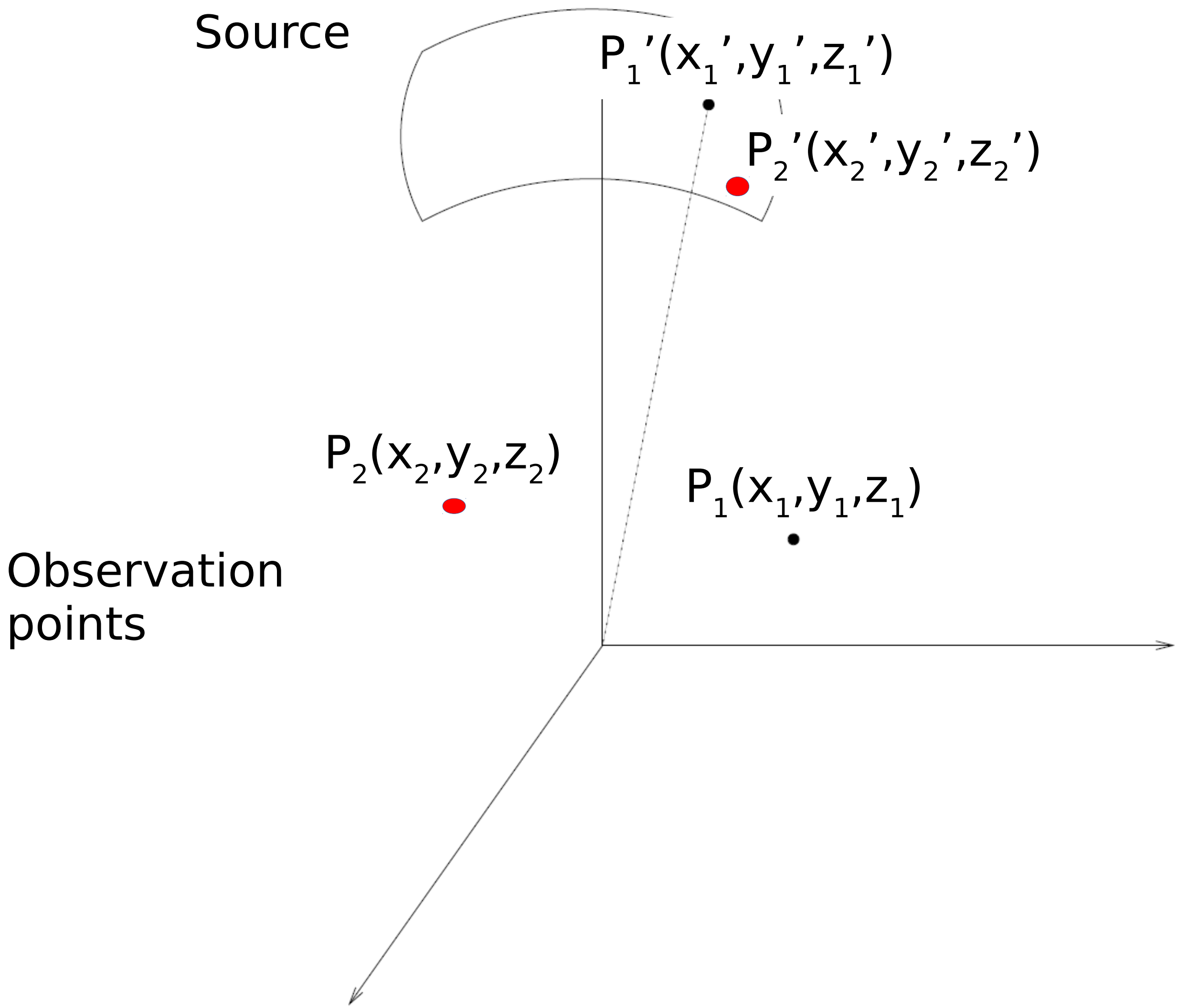
Van Cittert-Zernicke theorem

This relates the spatial coherence function, $V(r_1,r_2) = \langle E(r_1)E^*(r_2)\rangle$ to the intensity distribution of the incoming radiation $I(s)$. It shows that $V(r_1,r_2)$ only depends on $r_1 - r_2$ and if all the measurements are in a plane,

$$V(r_1,r_2) = F\{I(s)\}$$

Proof in "Principles of Optics" by Born and Wolf (Chapter 10).

Van Cittert-Zernicke theorem



Van Cittert-Zernicke theorem

$$\langle E(P_1)E^*(P_2)\rangle = \int I(l,m)e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_1-z_1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

$$u = (x_2 - x_1)/\lambda$$
$$v = (y_2 - y_1)/\lambda$$
$$w = (z_2 - z_1)/\lambda$$

$$V(u,v,w) = \int I(l,m)e^{-i2\pi[lu+mv+nw]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

Looks like a Fourier transform.

Spatial correlation of the electric field is related to the source brightness distribution.

Assumptions

Treated electric field like a scalar (was implicit when we used Huygen's principle).

Sources are far away (assume emission confined to "celestial sphere").

Celestial sphere is empty.

Radiation from astronomical sources is spatially incoherent.

Special cases

Observations are confined to the u-v plane, w = 0:

$$V(u,v) = \int \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-i2\pi[lu+mv]} dldm$$

Source brightness is limited to a small region of the sky -

$n = \sqrt{1-l^2-m^2} \simeq 1$

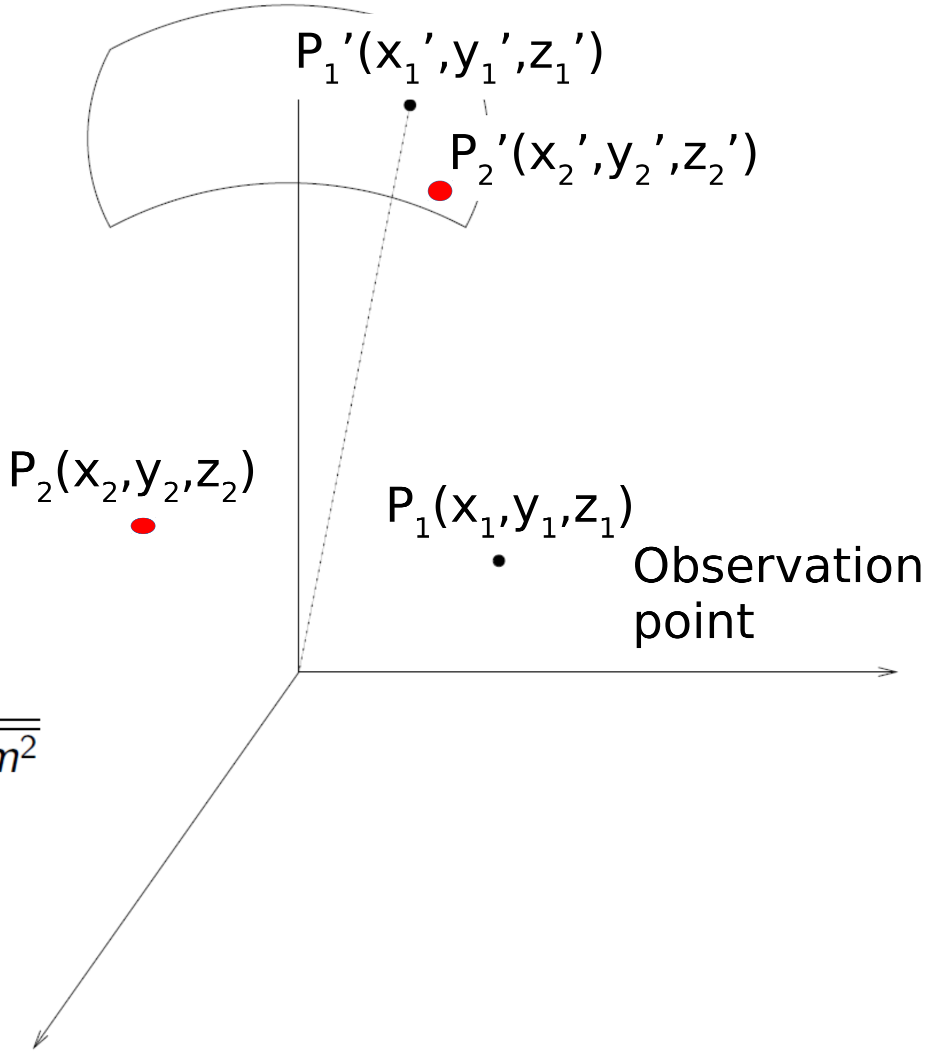
$$V(u,v,w) = e^{-i2\pi w} \int I(l,m) e^{-i2\pi[lu+mv]} dldm$$

Van Cittert-Zernicke theorem

According to van Cittert-Zernicke theorem: the source brightness distribution can be derived if one can measure the mutual coherence function of the electric fields.

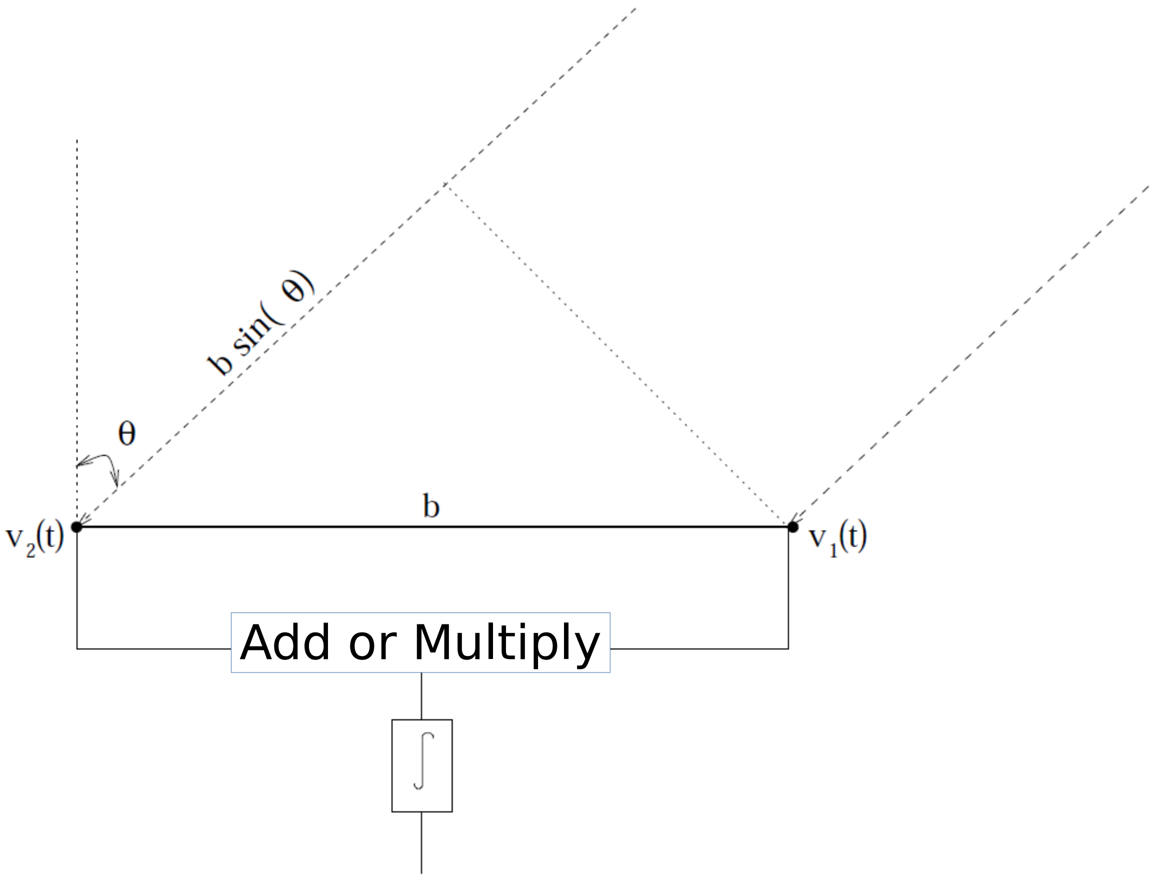
What is the dimension of the mutual coherence function ?

$$V(u,v,w) = \int I(l,m) e^{-i2\pi[lu+mv+nw]} \frac{dl dm}{\sqrt{1-l^2-m^2}}$$



A two element interferometer

Assume the radiation emitted by the source is monochromatic having a frequency ν

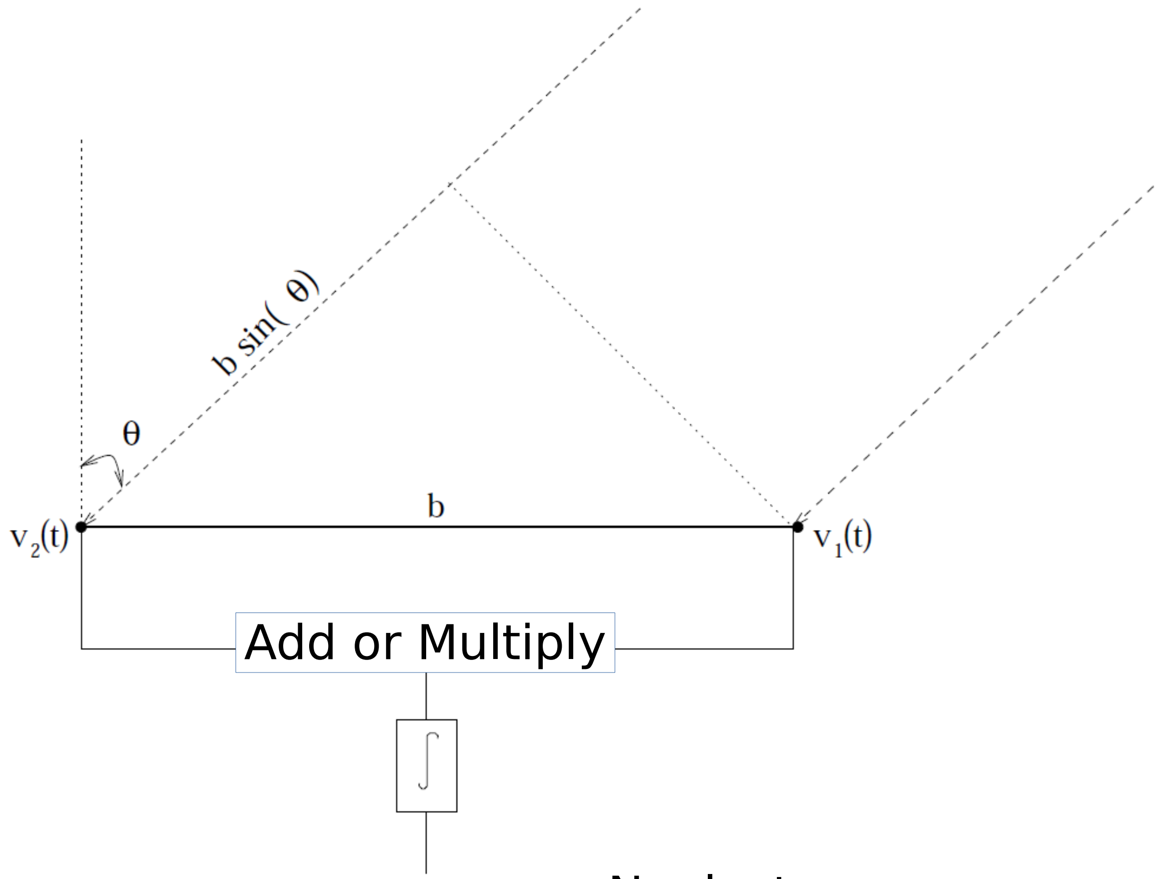


A two element interferometer

Assume the radiation emitted by the source is monochromatic having a frequency ν

The plane wave travels an extra distance to reach the second element – this is the *geometric delay*,

$$\tau_g = b \sin(\theta)/c$$



Neglect instrumental delay for the moment

A two element interferometer

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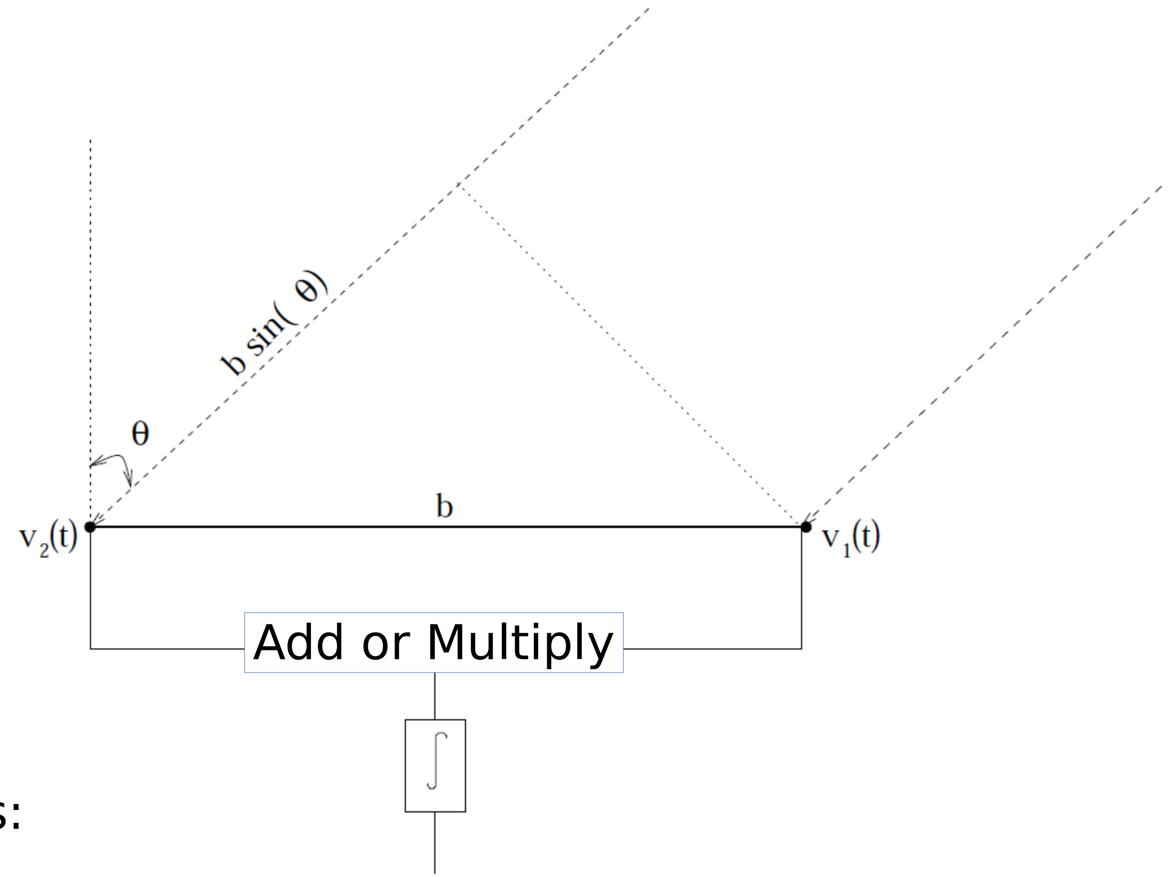

The voltages at the two points:

$$v_1(t) = cos(2\pi\nu t) \quad \text{and} \quad v_2(t) = cos(2\pi\nu(t - \tau_g))$$

A two element interferometer: adding

$$[v_1(t) + v_2(t)]^2 = [cos(2\pi\nu t) + cos(2\pi\nu(t - \tau_g))]^2$$

Squaring and then reducing the RHS using trigonometric identities and averaging:

$$\langle[v_1(t) + v_2(t)]^2\rangle = 1 + cos(2\pi\nu\tau)$$

$$= 1 + cos(2\pi\frac{b}{\lambda}sin(\theta))$$

The offset term : have to detect over and above the offset term that is dominated by noise that also varies and makes detection of sources difficult.
We will discuss multiplying interferometers henceforth.

A two element interferometer: multiplying

Assuming averaging time is much longer than 1/ν

$$r(\tau_g) = \frac{1}{\mathrm{T}} \int_{t-T/2}^{t+T/2} \cos(2\pi\nu t)\cos(2\pi\nu(t-\tau_g))dt$$

$$= \frac{1}{\mathrm{T}} \int_{t-T/2}^{t+T/2} (\cos(4\pi\nu t - 2\pi\nu\tau_g) + \cos(2\pi\nu\tau_g))dt$$

Work out the algebra

A two element interferometer: multiplying

Assuming averaging time is much longer than 1/ν

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Work out the algebra

$\cos(4\pi\nu t)$

Averages out to zero

A two element interferometer: multiplying

Assuming averaging time is much longer than 1/ν

$$r(\tau_g) = \frac{1}{\mathrm{T}}\int_{t-T/2}^{t+T/2} \cos(2\pi\nu t)\cos(2\pi\nu(t-\tau_g))dt$$

Work out the algebra

$$= \frac{1}{\mathrm{T}}\int_{t-T/2}^{t+T/2} (\cos(4\pi\nu t - 2\pi\nu\tau_g) + \cos(2\pi\nu\tau_g))dt$$

$\cos(4\pi\nu t)$

Averages out to zero

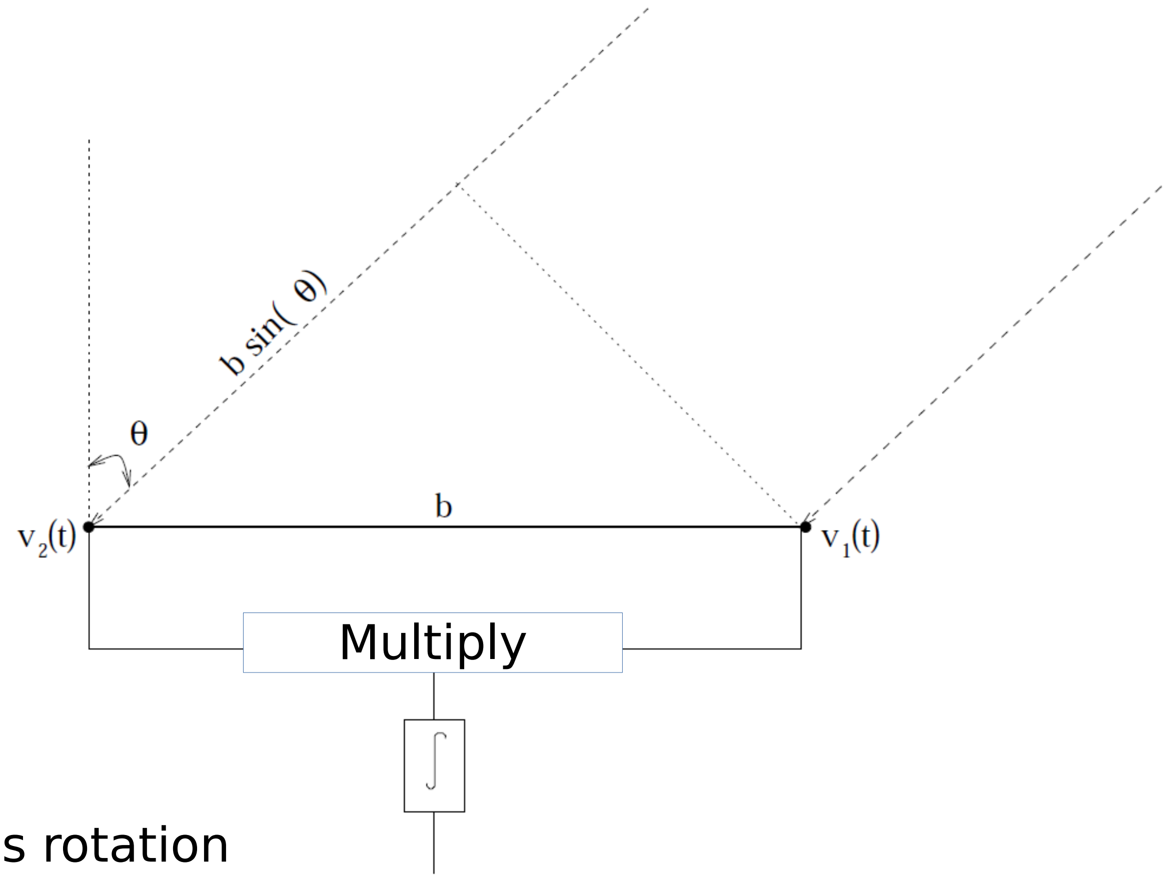
$$= \cos(2\pi\nu\tau_g)$$

A two element interferometer: multiplying

τ_g = b sin(θ)/c

The theta changes with source rise and set. Assuming exactly east-west baseline vector, and source at declination 0 deg,

$$\theta = \Omega_E t$$



Ω_E angular frequency of Earth's rotation

= 7.29 x 10^{-5} rad/s

How many "/s is that ?

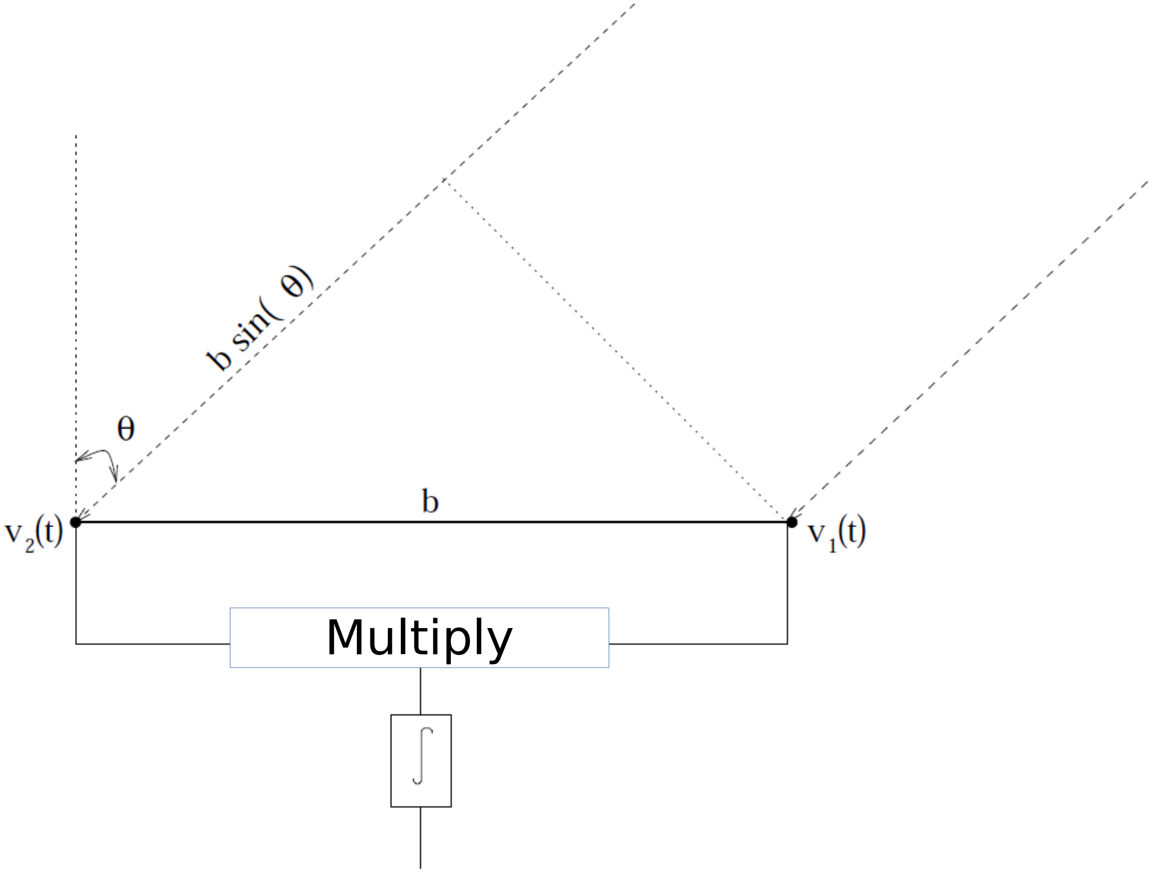
A two element interferometer: multiplying

τ_g = b sin(θ)/c

The theta changes with source rise and set. Assuming exactly east-west baseline vector, and source at declination 0 deg,

$$\theta = \Omega_E t$$

$$r(\tau_g) = \cos(2\pi\nu\tau_g)$$



A two element interferometer: multiplying

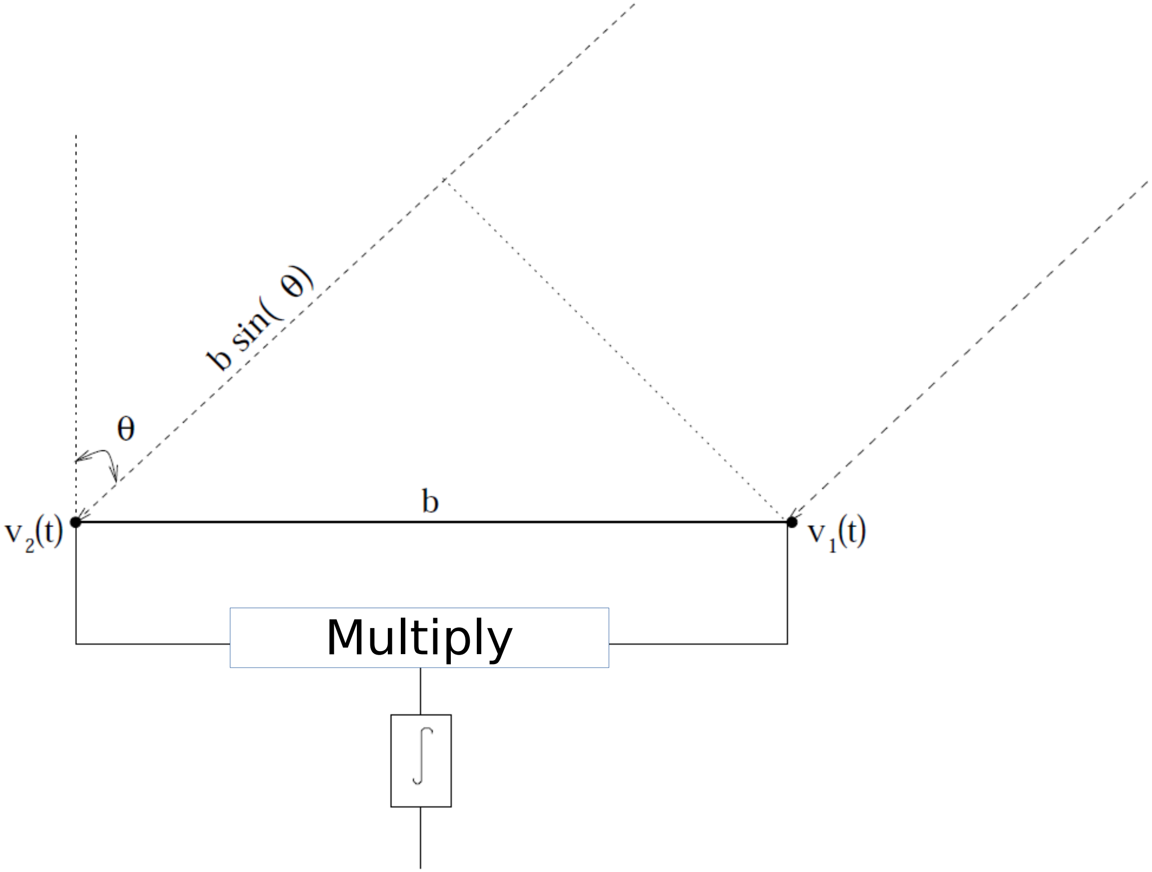
τ_g = b sin(θ)/c

The theta changes with source rise and set. Assuming exactly east-west baseline vector, and source at declination 0 deg,

$$\theta = \Omega_E t$$

$$r(\tau_g) = \cos(2\pi\nu\tau_g)$$

$$r(\tau_g) = \cos(2\pi\nu \times b/c \times \sin(\Omega_E(t - t_z)))$$

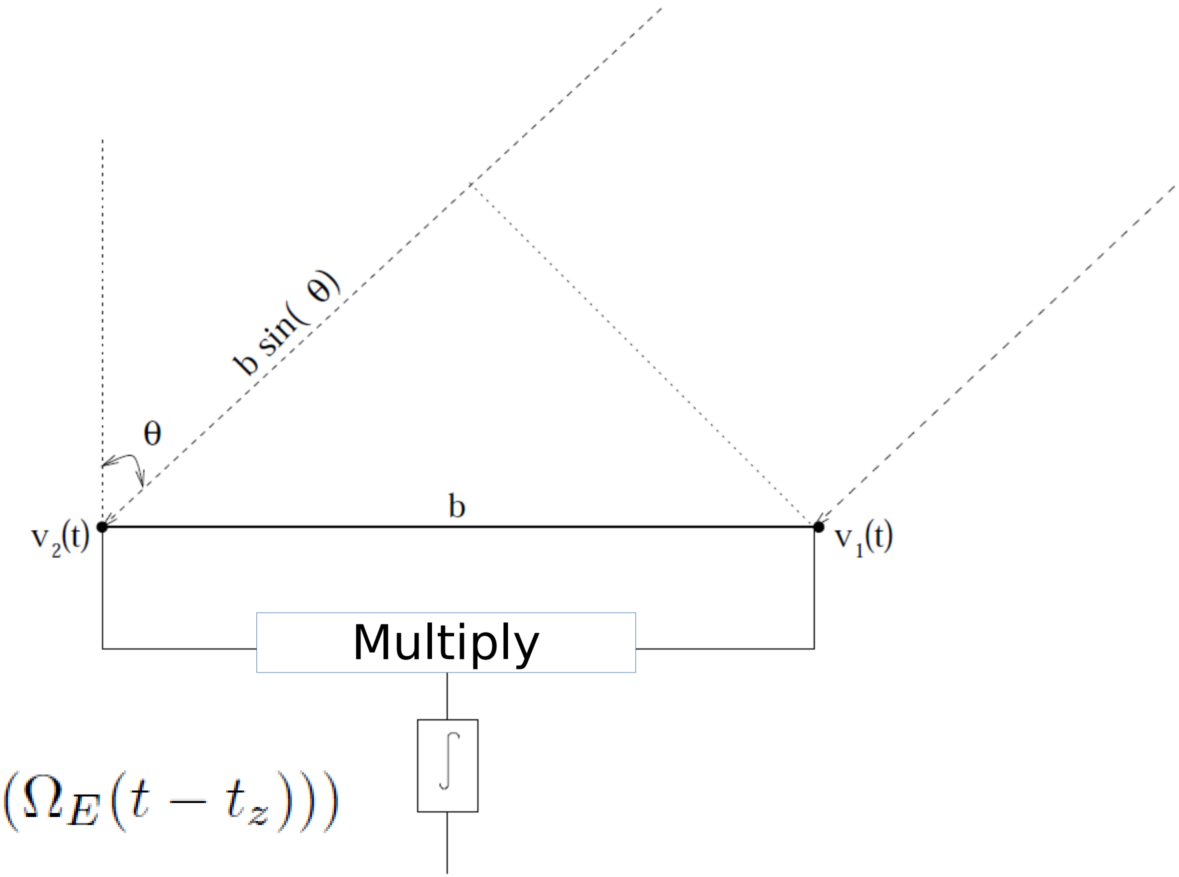


t_z is the time when the source is at the zenith.

A two element interferometer: multiplying

τ_g = b sin(θ)/c

The theta changes with source rise and set. Assuming exactly east-west baseline vector, and source at declination 0 deg,



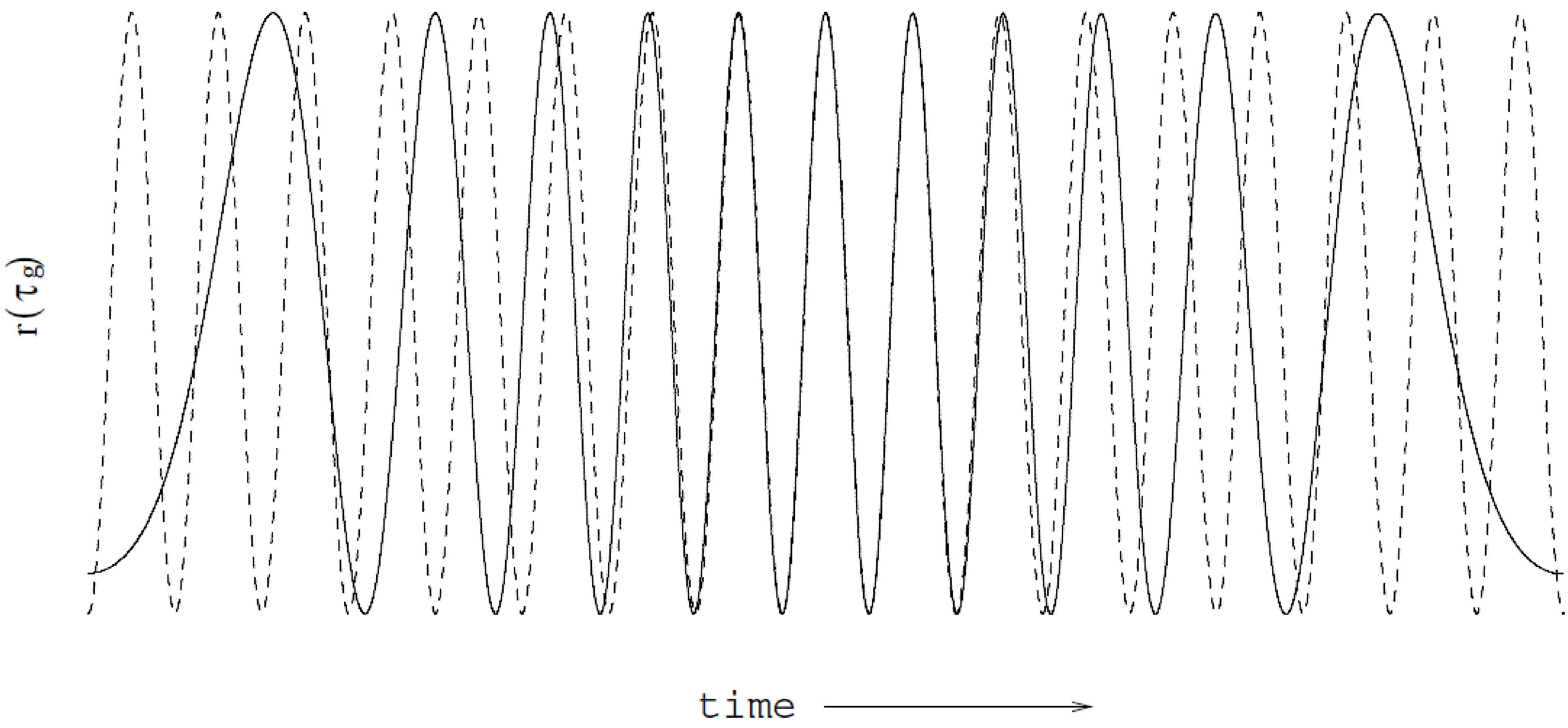
$$\theta = \Omega_E t$$

$$r(\tau_g) = \cos(2\pi\nu \ \times \ b/c \ \times \ \sin(\Omega_E(t - t_z)))$$

$r(\tau_g)$ *is called the fringe.*

Fringe

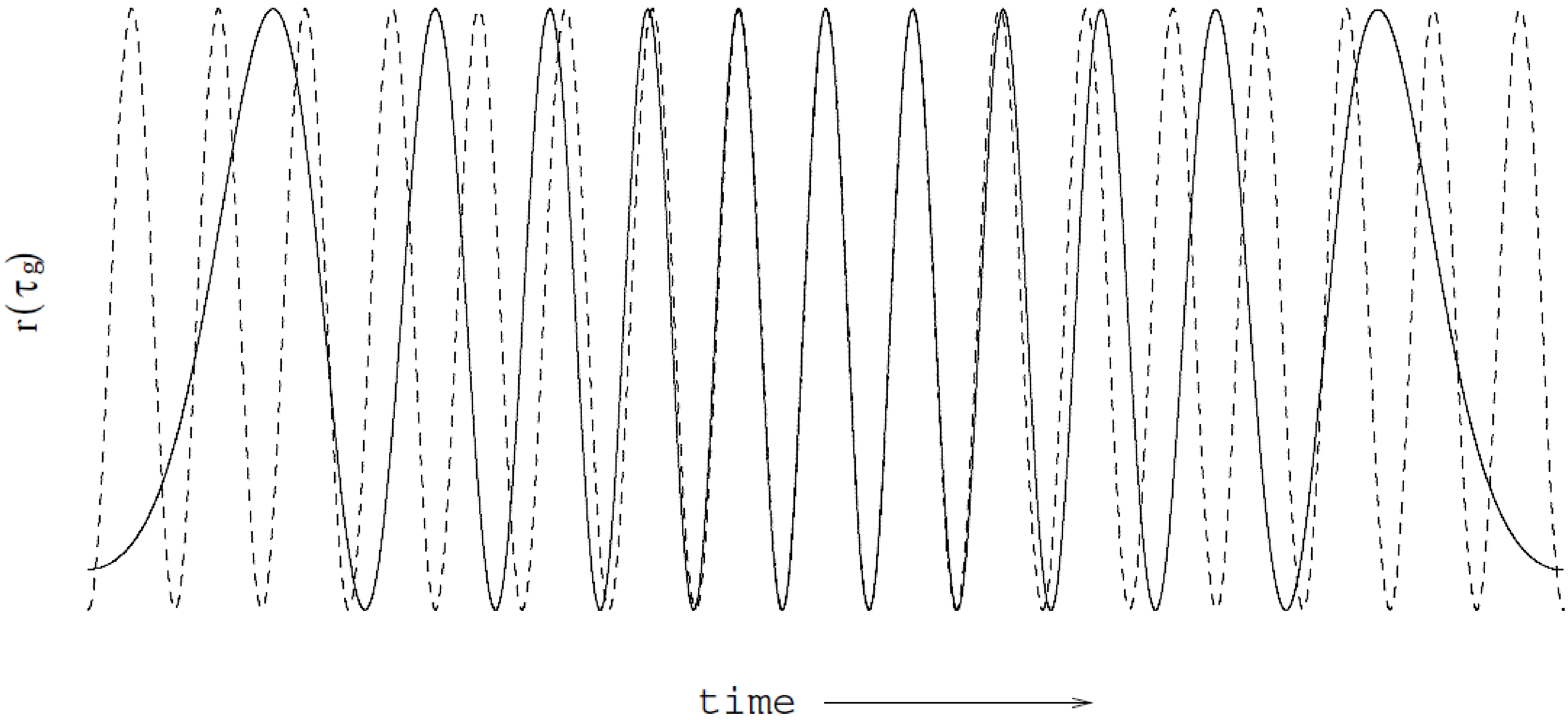
$$r(\tau_g) = \cos(2\pi\nu \times b/c \times \sin(\Omega_E(t - t_z)))$$



Solid line: observed output
Dashed line: pure sinusoid with frequency equal to the maximum instantaneous frequency of the fringe.

Fringe

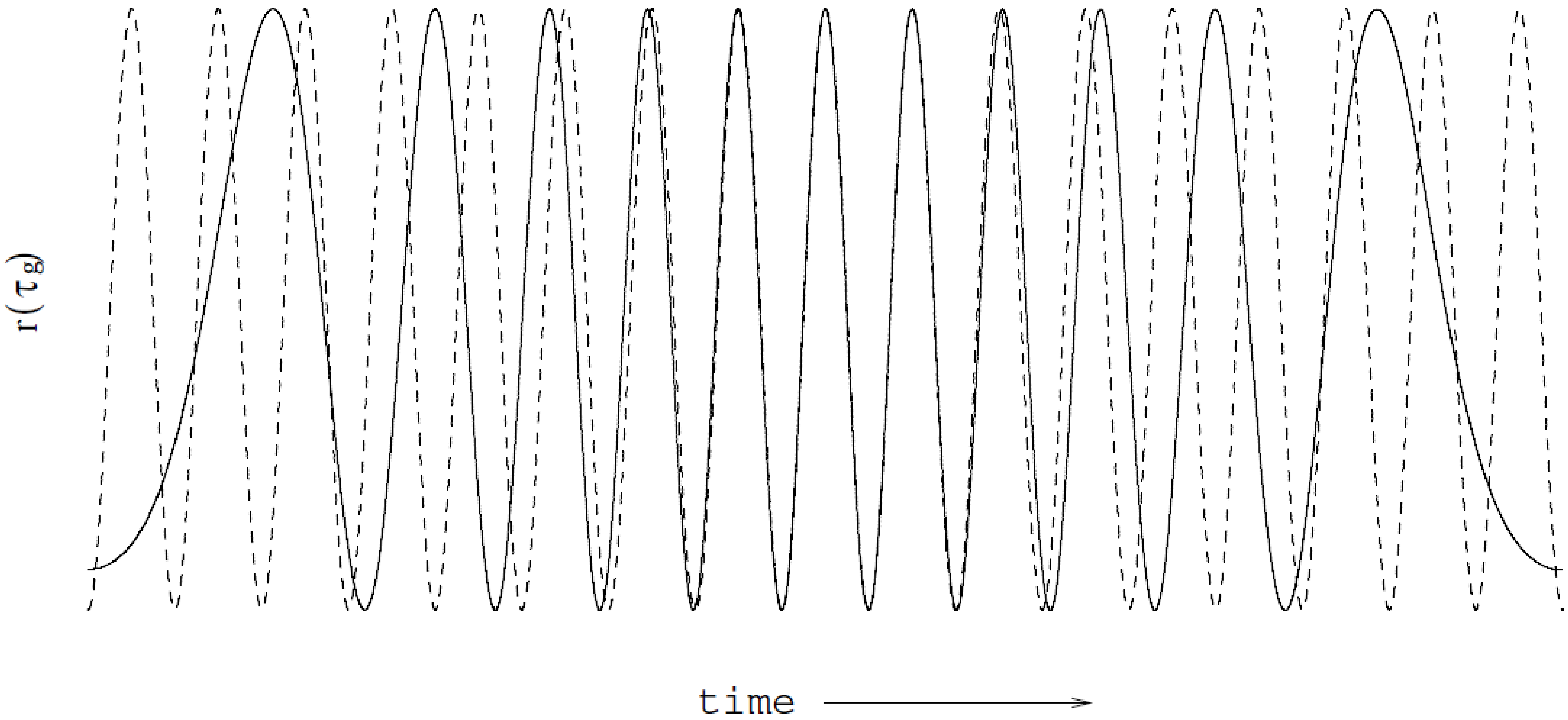
$$r(\tau_g) = \cos(2\pi\nu \times b/c \times \sin(\Omega_E(t - t_z)))$$



If the RA is known the time when the “fringe frequency” will peak can be predicted. Thus between measured fringe frequency peak and that expected one can accurately find the position of the source.

Fringe

$$r(\tau_g) = \cos(2\pi\nu \ \times \ b/c \ \times \ \sin(\Omega_E(t - t_z)))$$



For a point source the fringe amplitude will remain the same. But if extended then the fringe amplitude will decrease due to waves arriving at a slightly different path differences from different parts of the source. For a very large source the fringe amplitude will be zero: source is *“resolved out”*.

Resolution $r(\tau_g) = \cos(2\pi\nu \times b/c \times \sin(\Omega_E(t - t_z)))$

Sources smaller than the fringe spacings will all appear as point sources. When the source size is such that the waves from different parts of the source give rise to the same phase lags, then the source will appear as a point source.

When the source size is such that the waves coming from different parts of the source give rise to the same phase lags (within a factor smaller than π), then the source will appear as a point source.

Resolution $r(\tau_g) = \cos(2\pi\nu \times b/c \times \sin(\Omega_E(t - t_z)))$

Sources smaller than the fringe spacings will all appear as point sources. When the source size is such that the waves from different parts of the source give rise to the same phase lags, then the source will appear as a point source.

The minimum source size that can be resolved by the interferometer:

$$\pi\nu\Delta\theta b/c \lesssim \pi \implies \Delta\theta \lesssim \lambda/b$$

The resolution of a two element interferometer with baseline length b is $\sim \lambda/b$

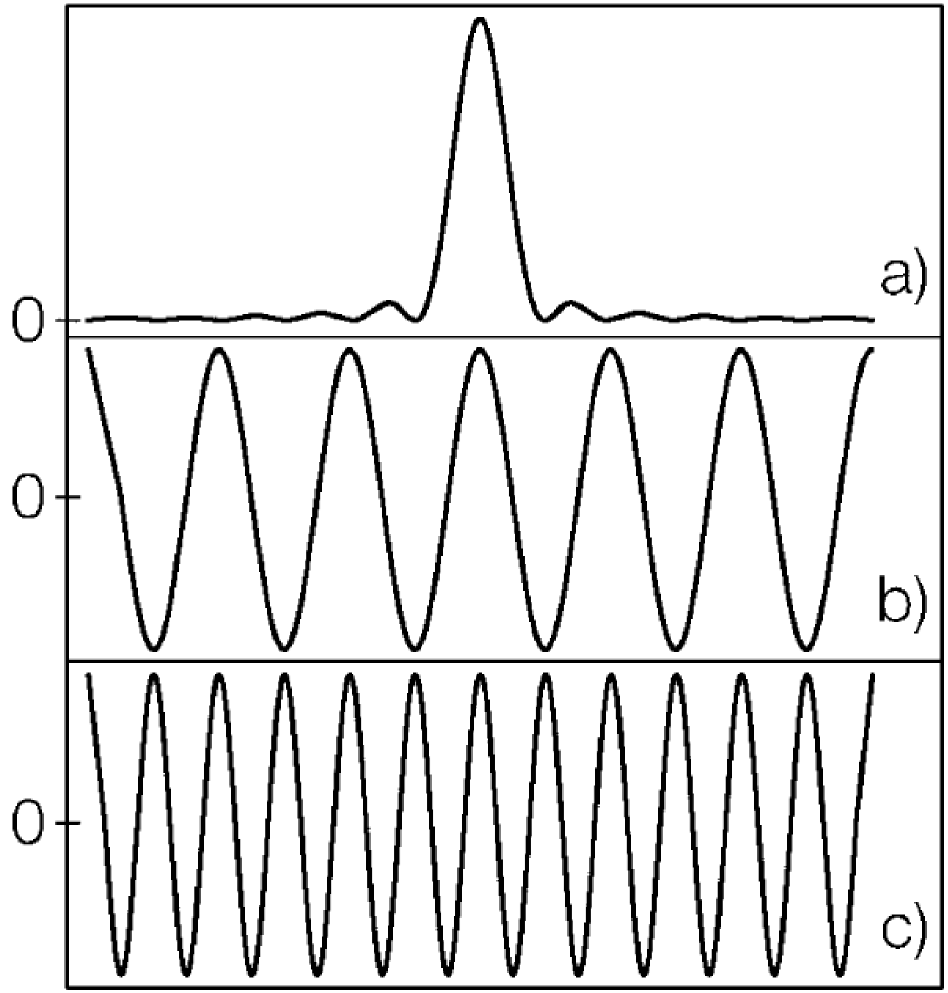
Larger the b, higher will be the resolution.

Power patterns

A uniformly illuminated aperture of diameter D

A two element multiplying interferometer each element of diameter d and spacing D where d<<D

Spacing of 2D



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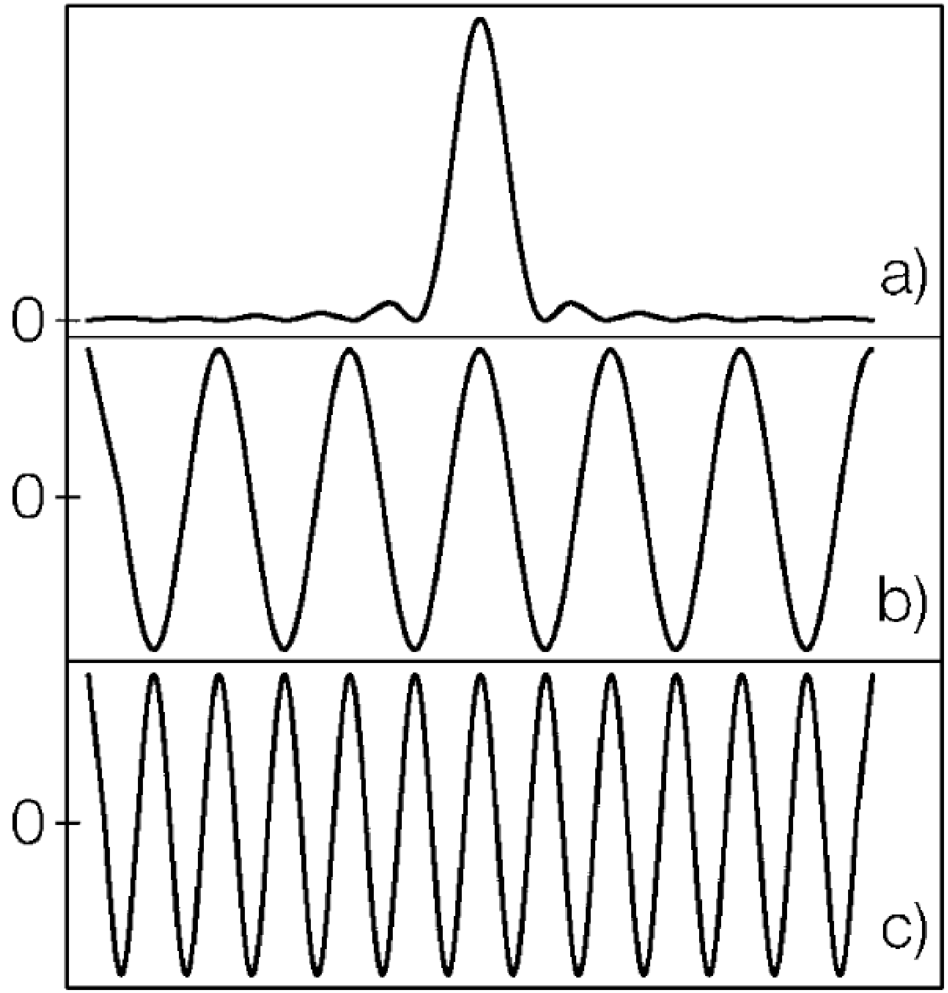


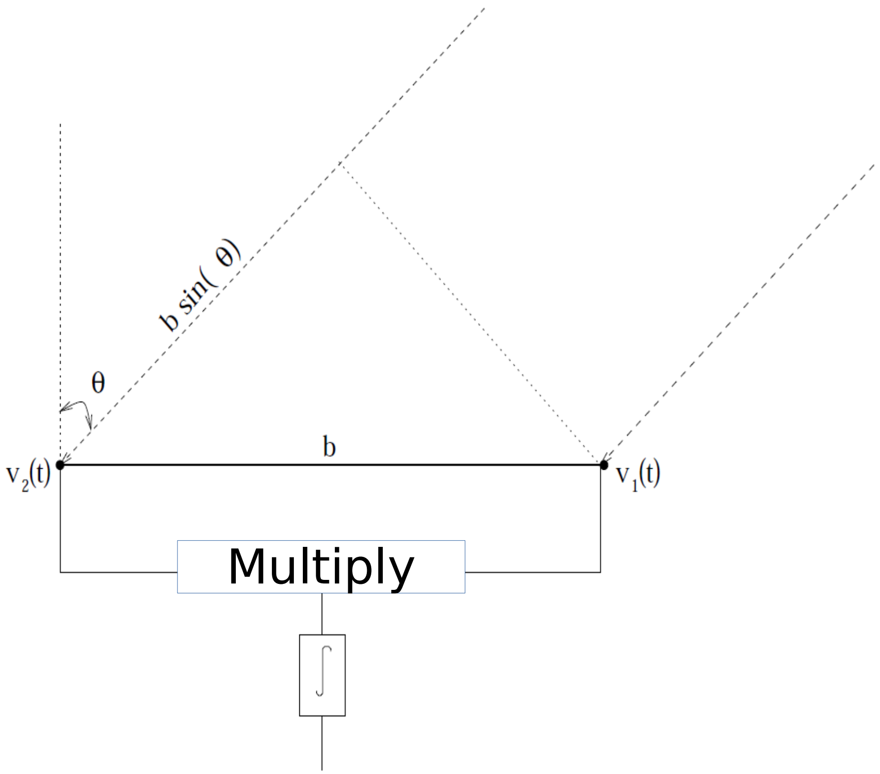
NOTE: Collecting area smaller than single aperture

A two element interferometer

One can infer the source position and size with a two element interferometer.

If we make measurements by varying the baseline length and orientations one will get different constraints on the source size and source brightness.

Using van Cittert Zernike theorem one can then infer the correct source brightness distribution on the sky.



Quasi-mono-chromatic waves

In reality we have waves coming from a band $\Delta\nu$ around ν.

Radiation if at one frequency arrives in phase, at an adjacent frequency it will be out of phase and for a large enough separation in frequencies, they may be 180 deg out of phase. Thus averaging all these together will decrease the amplitude of the fringe.

$$
\begin{aligned}
r(\tau_g) &= \frac{1}{\Delta\nu}\int_{\nu-\frac{\Delta\nu}{2}}^{\nu+\frac{\Delta\nu}{2}} \cos(2\pi\nu\tau_g)d\nu \\
&= \frac{1}{\Delta\nu} Re\left[\int_{\nu-\frac{\Delta\nu}{2}}^{\nu+\frac{\Delta\nu}{2}} e^{i2\pi\nu\tau_g} d\nu\right] \\
&= \cos(2\pi\nu\tau_g)\left[\frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}\right]
\end{aligned}
$$

Quasi-mono-chromatic waves

In reality we have waves coming from a band $\Delta\nu$ around ν.

Radiation if at one frequency arrives in phase, at an adjacent frequency it will be out of phase and for a large enough separation in frequencies, they may be 180 deg out of phase. Thus averaging all these together will decrease the amplitude of the fringe.

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r(\tau_g) &= \frac{1}{\Delta\nu}\int_{\nu-\frac{\Delta\nu}{2}}^{\nu+\frac{\Delta\nu}{2}} \cos(2\pi\nu\tau_g)d\nu \\
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\end{aligned}
$$

Sinc function here decreases rapidly as the bandwidth increases. Termed as “fringe washing”.

Need a way to average over large bandwidths without losing the fringe amplitude.

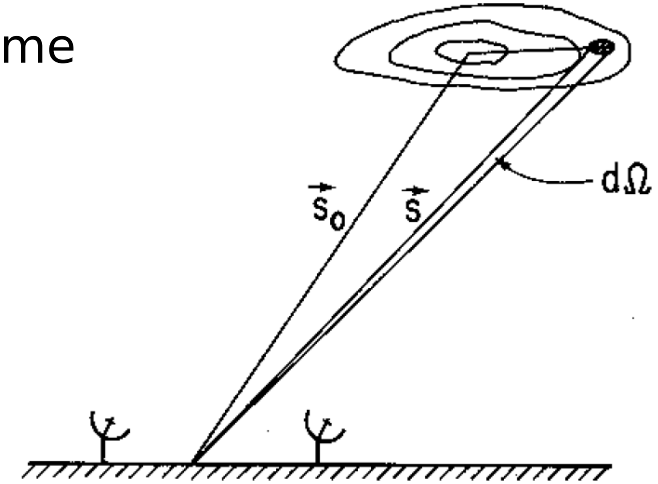
Extended source

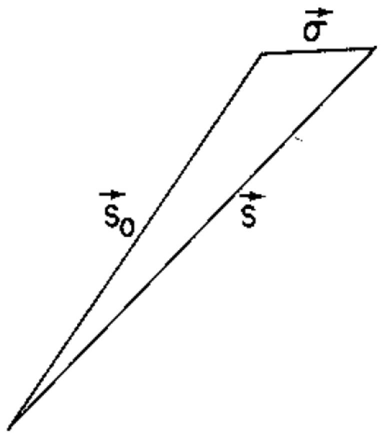
Consider a source at s_0 with some small extent. Any point on the source can be written as

$s = s_0 + \sigma$

$s_0 . \sigma = 0$

$\tau_g = s_0 . b$





Extended source

Consider a source at s_0 with some small extent. Any point on the source can be written as

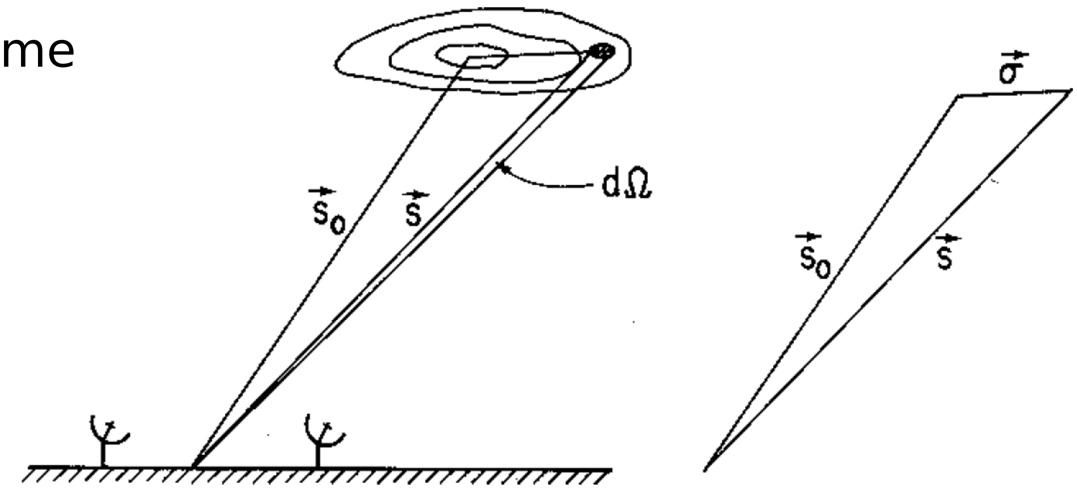
$s = s_0 + \sigma$

$s_0 . \sigma = 0$

$\tau_g = s_0.b$

From van Cittert Zernike theorem:

$$r(\tau_g) \quad = \quad Re\left[\int I(\mathbf{s})e^{\frac{-i2\pi \mathbf{s}.\mathbf{b}}{\lambda}} d\mathbf{s}\right]$$



$s.b = s_0.b + \sigma.b$

Extended source

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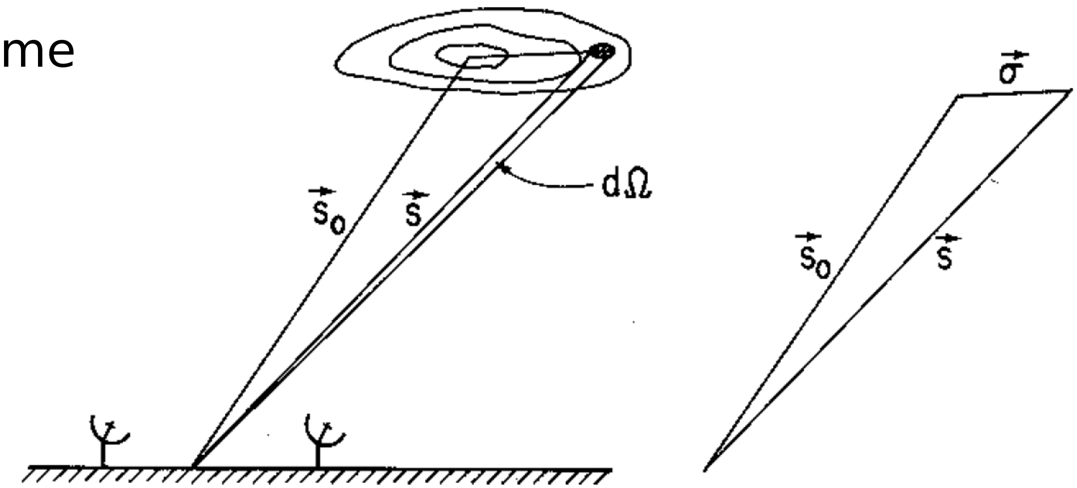

$$
\begin{aligned}
r(\tau_g) &= Re\left[\int I(\mathbf{s}) e^{\frac{-i2\pi \mathbf{s}.\mathbf{b}}{\lambda}} d\mathbf{s}\right] \\
&= Re\left[e^{\frac{-i2\pi \mathbf{s_0}.\mathbf{b}}{\lambda}} \int I(\mathbf{s}) e^{\frac{-i2\pi \sigma.\mathbf{b}}{\lambda}} d\mathbf{s}\right]
\end{aligned}
$$

$s.b = s_0.b + \sigma.b$

Extended source

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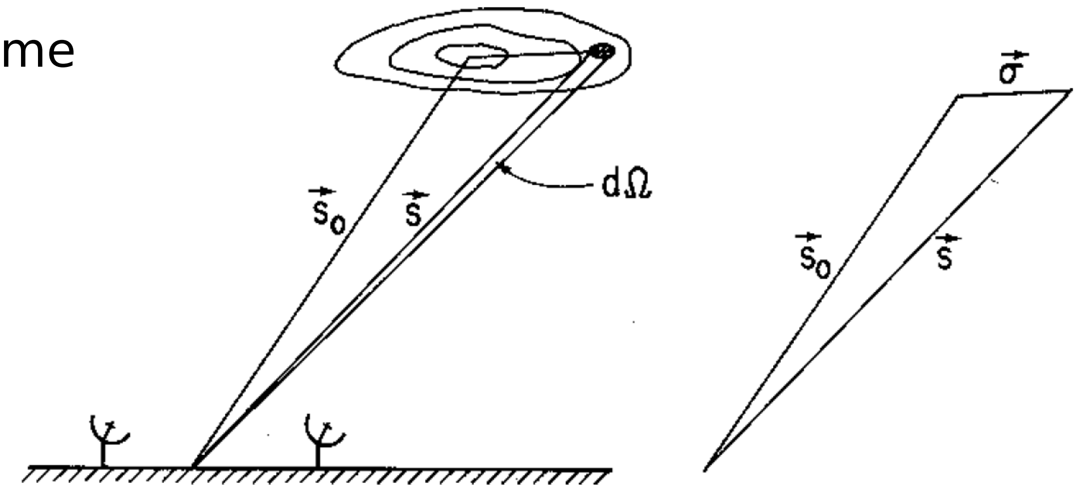


$$r(\tau_g) = Re\left[\int I(\mathbf{s}) e^{\frac{-i2\pi \mathbf{s}.\mathbf{b}}{\lambda}} d\mathbf{s}\right]$$

$s.b = s_0.b + \sigma.b$

$$= Re\left[e^{\frac{-i2\pi \mathbf{s_0}.\mathbf{b}}{\lambda}} \int I(\mathbf{s}) e^{\frac{-i2\pi \sigma.\mathbf{b}}{\lambda}} d\mathbf{s}\right]$$

$$= |\mathcal{V}| \cos(2\pi\nu\tau_g + \Phi_{\mathcal{V}})$$

where $\mathcal{V} = |\mathcal{V}| e^{-i\Phi_{\mathcal{V}}}$

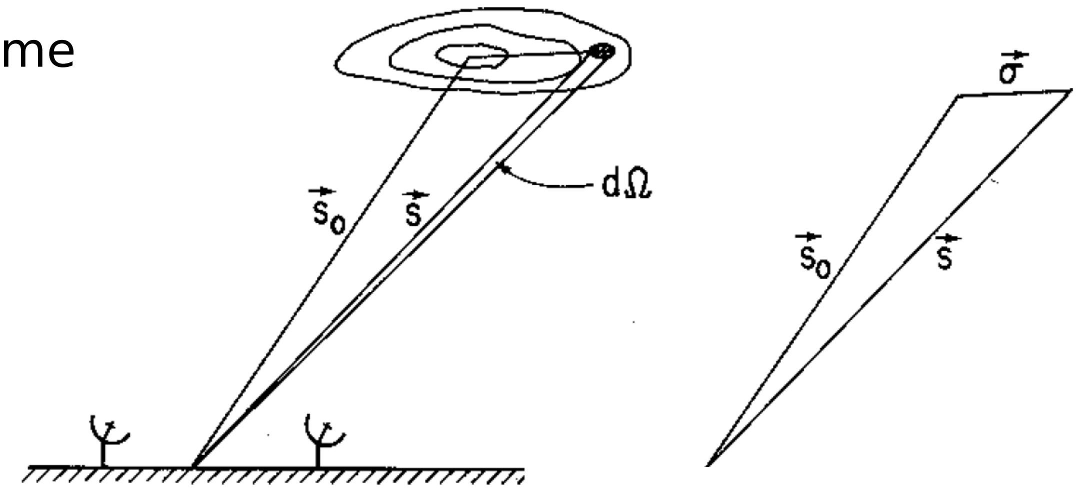
Extended source

Consider a source at s_0 with some small extent. Any point on the source can be written as

$s = s_0 + \sigma$

$s_0 . \sigma = 0$

$\tau_g = s_0 . b$



$$r(\tau_g) = |\mathcal{V}| \cos(2\pi\nu\tau_g + \Phi_{\mathcal{V}})$$

where

$$\mathcal{V} = |\mathcal{V}| e^{-i\Phi_{\mathcal{V}}}$$

Only contains the variation of the fringe as a function of earth's rotation or source rise-set. If an equal delay is introduced in the signals' path we will have:

$$r(\tau_g) = |\mathcal{V}| \cos(\Phi_{\mathcal{V}})$$

This instrumental delay has to change continuously as τ_g changes: *delay tracking*

Extended source

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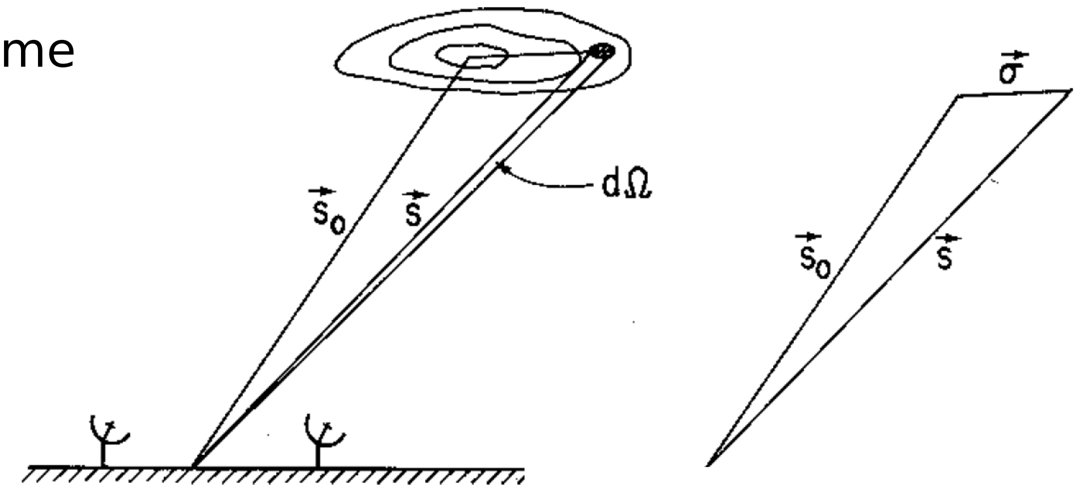


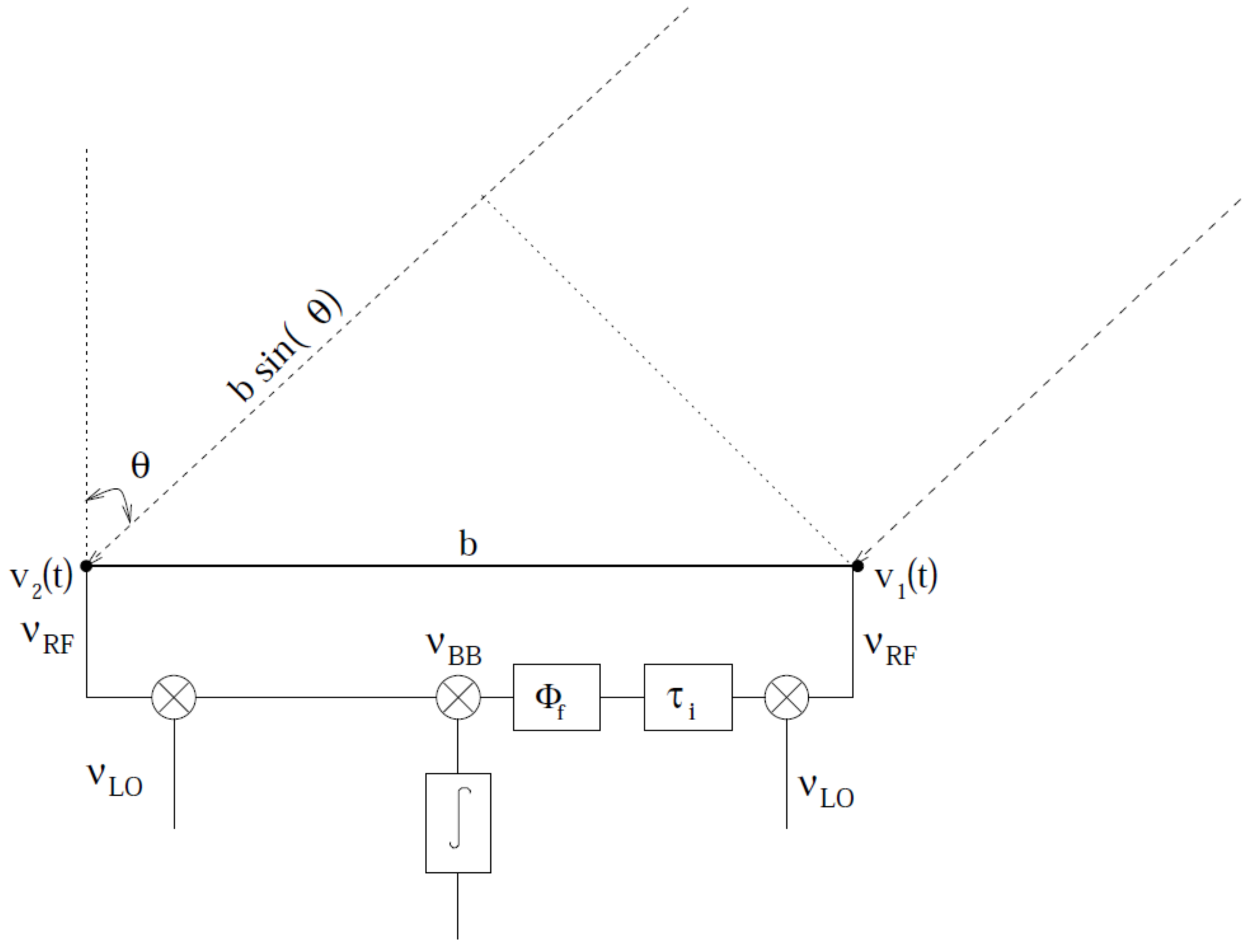
$$r(\tau_g) \ = \ |\mathcal{V}| \cos(2\pi\nu\tau_g + \Phi_{\mathcal{V}})$$ where $$\mathcal{V} = |\mathcal{V}| e^{-i\Phi_{\mathcal{V}}}$$

Only contains the variation of the fringe as a function of earth's rotation or source rise-set. If an equal delay is introduced in the signals' path we will have:

$$r(\tau_g) = |\mathcal{V}| \cos(\Phi_{\mathcal{V}})$$

While τ_g is in RF the delay tracking is in baseband and thus needs to be properly accounted.

Two element interferometer in practice



Delay tracking and fringe stopping

$$\nu_{LO} = \nu_{RF} - \nu_{BB}$$

$$\tau_i = \tau_g + \Delta\tau$$

$$\begin{aligned} r(\tau_g) &= |\mathcal{V}| \langle \cos(\Phi_{\mathcal{V}} + 2\pi\nu_{BB}t - 2\pi\nu_{RF}\tau_g) \cos(2\pi\nu_{BB}(t - \tau_i) + \Phi_f) \rangle \\ &= |\mathcal{V}| cos(\Phi_{\mathcal{V}} + 2\pi(\nu_{RF} - \nu_{BB})\tau_g - \nu_{BB}\Delta\tau - \Phi_f) \\ &= |\mathcal{V}| \cos(\Phi_{\mathcal{V}} + 2\pi\nu_{LO}\tau_g - \nu_{BB}\Delta\tau - \Phi_f) \end{aligned}$$

$$\Phi_f = 2\pi\nu_{LO}\tau_g$$