- Self-calibration
- Sensitivity
- W-term

# Astronomical Techniques II : Lecture 13

**Ruta Kale**

Low Frequency Radio Astronomy (Chp. 14)
http://www.ncra.tifr.res.in/ncra/gmrt/gmrt-users/low-frequency-radio-astronomy

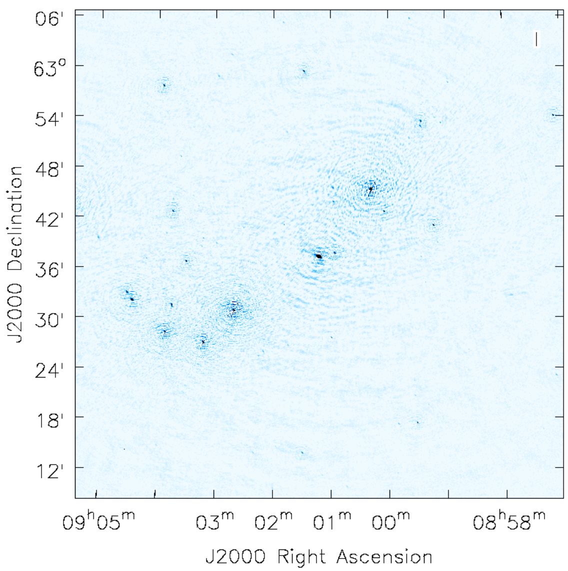
Synthesis imaging in radio astronomy II, Chp 9, 10, 19, 31, 32

Interferometry and synthesis is radio astronomy, Chp. 12

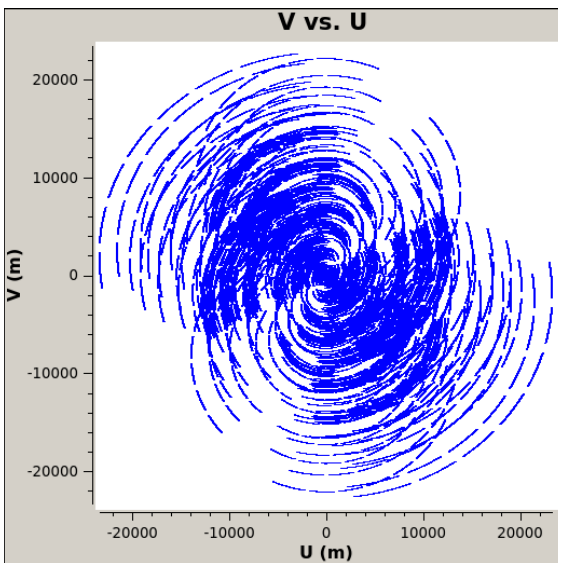
Other references are given on the slides.

# Recall: Imaging

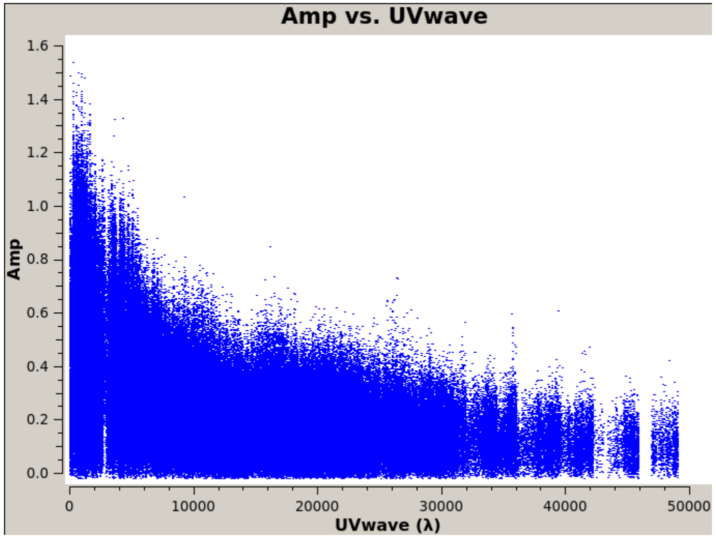
$$I^D(l,m) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(u,v)V'(u,v)e^{2\pi i(ul+vm)}\,du\,dv$$




Image




Sampling




Visibilities (complex numbers)
Only amp. shown here

# Recall: Imaging → Deconvolution

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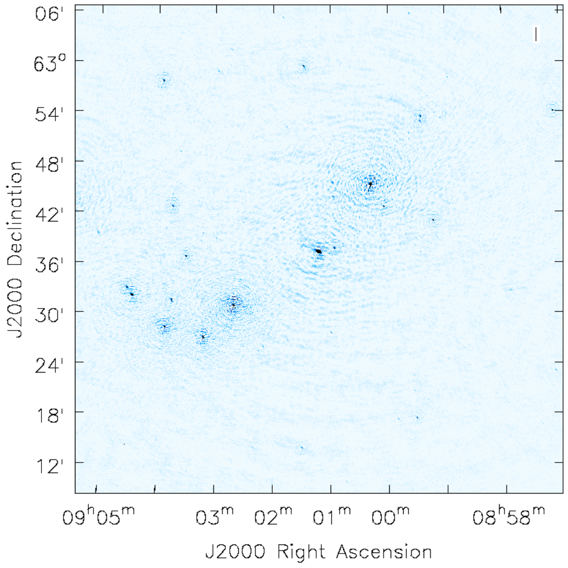
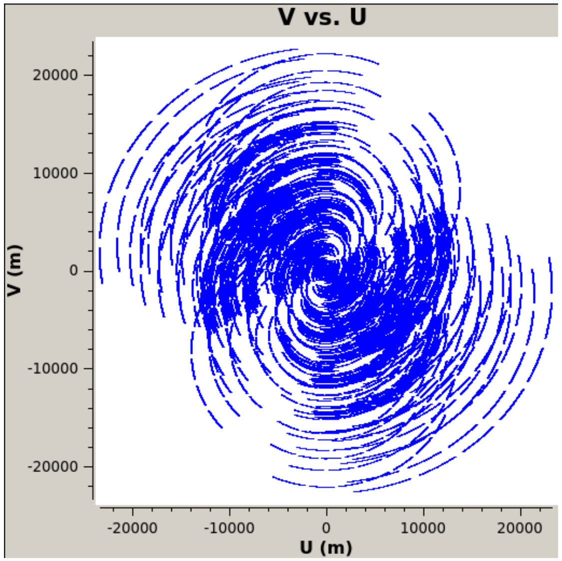
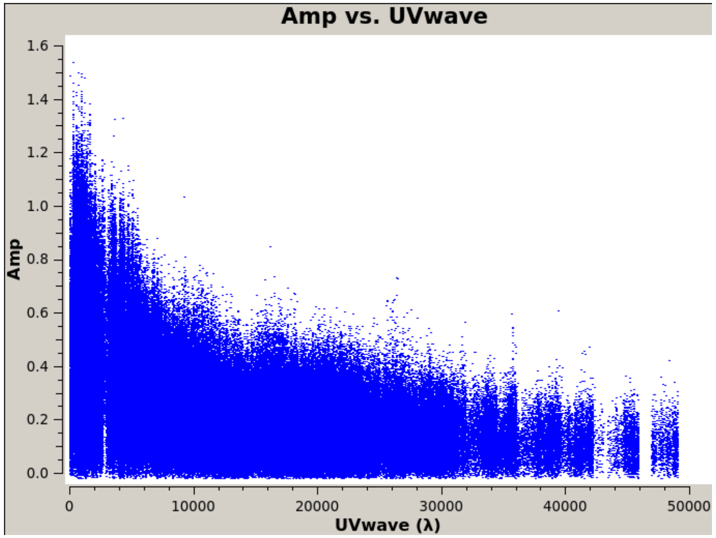



Image




Sampling




Visibilities (complex numbers)
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# Self-calibration

Aim is to produce a model of the intensity distribution, the Fourier transform of which when corrected by gain factors will reproduce the measured visibilities within the noise level.

A convenient method due to Schwab 1980 is to *minimize* the sum of squares of residuals by varying complex gains $g_i$, $g_j$ and the model sky,

$$\mathcal{S} = \sum_k \sum_{\substack{i,j \\ i \neq j}} w_{ij}(t_k) \left| \widetilde{V}_{ij}(t_k) - g_i(t_k) g_j^*(t_k) \widehat{V}_{ij}(t_k) \right|^2$$

weights — Observed visibilities — Complex gains — Model visibilities

The time over which gains are assumed to be constant depend on the effects that govern their variation.

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In most cases we have small number of degrees of freedom (the gains to be determined) and a large number of measurements of visibilities.

# Self-calibration

The sky intensity model can be iteratively refined – this is done via self-calibration. The name follows after the fact that we are using the image itself as its own calibrator.

The iterative recipe is:

- Make an initial model of the sky (use CLEAN).
- Solve for complex gains.
- find the corrected visibility,

$$V_{ij,\mathrm{corr}}(t) = \frac{\widetilde{V}_{ij}(t)}{g_i(t)g_j^*(t)}$$

- Form a new model from the corrected data using constraints on the source structure.
- Again solve for complex gains and repeat until there is no improvement.

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Self-calibration is found to work. But there is no proof of convergence.
- for telescopes with large number of elements there are few variables in terms of gains as compared to the available constraints.
- sources are “simple” in their structure.

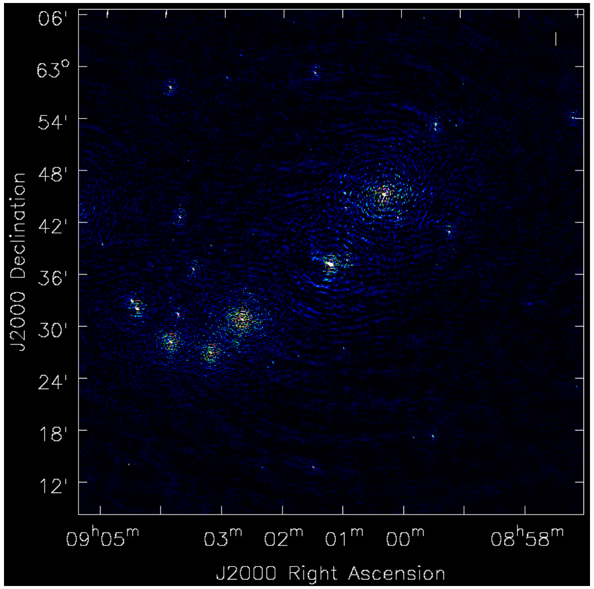
# Self-calibration (caution !)

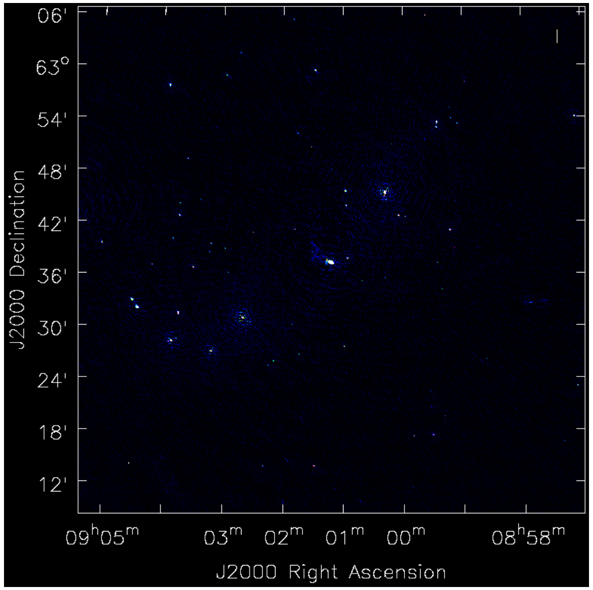
Self-calibration is found to work. But there is no proof of convergence.
 - for telescopes with large number of elements there are few variables in terms of gains as compared to the available constraints.
- sources are “simple” in their structure.

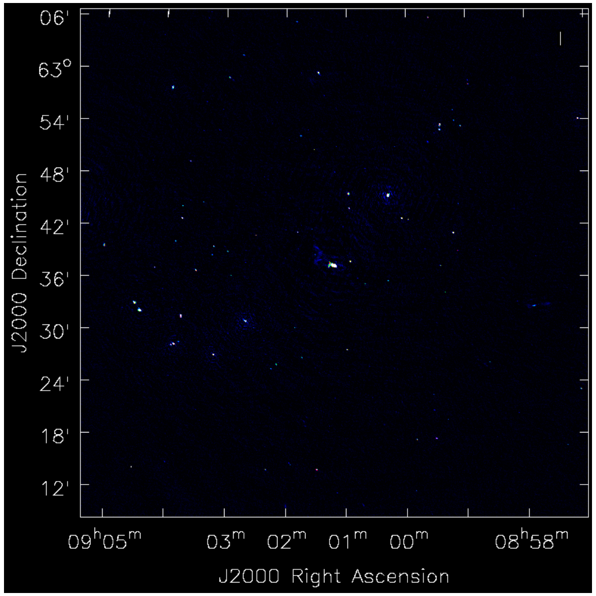
Can lead to totally wrong results if the model incorporates features from the image which are due to errors in calibration – the very effect which this procedure is trying to remove.

Solved for gain phases-only in the first few iterations and then for both amplitude and phase of the complex gain.

# Example of self-calibration



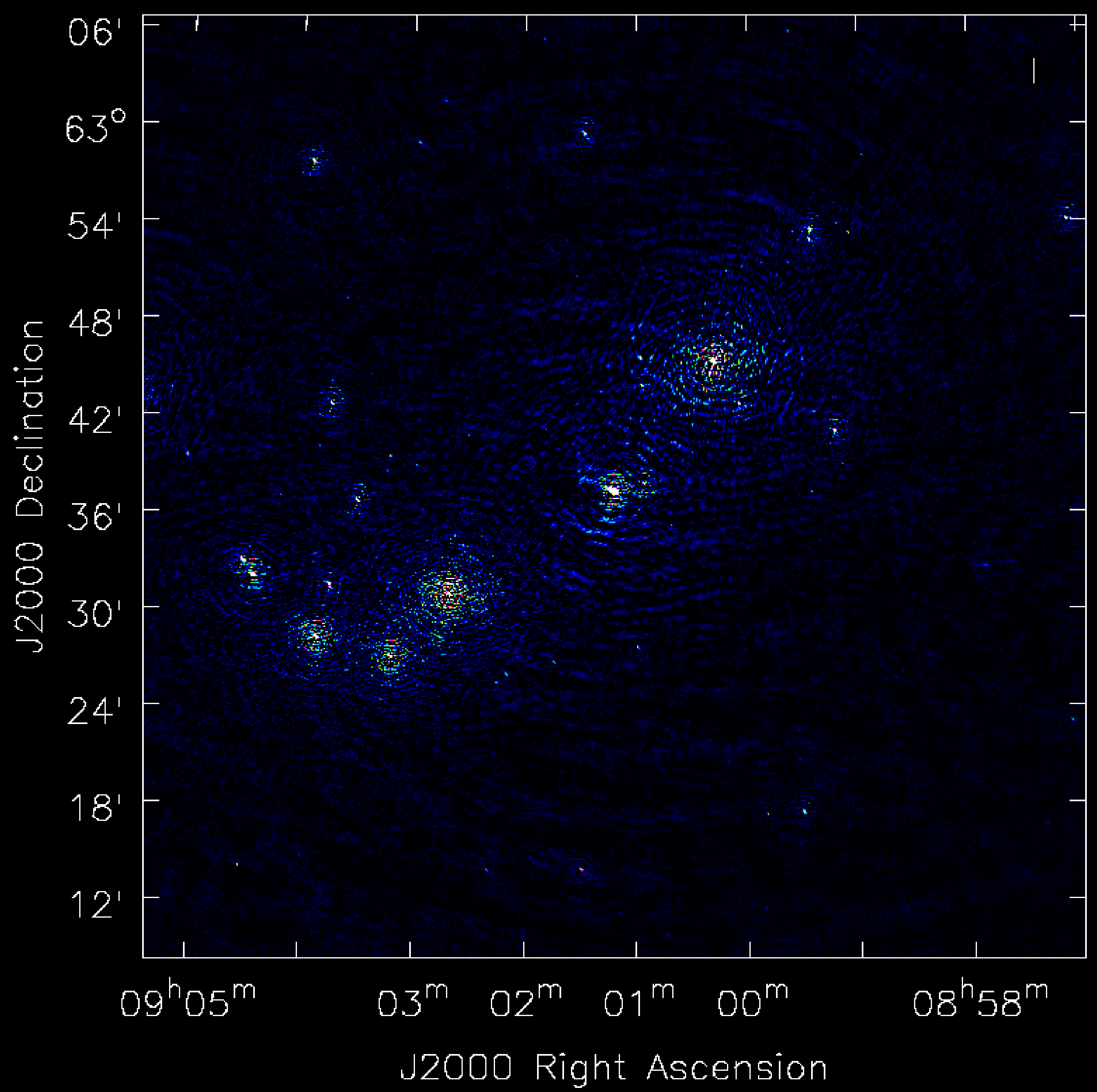



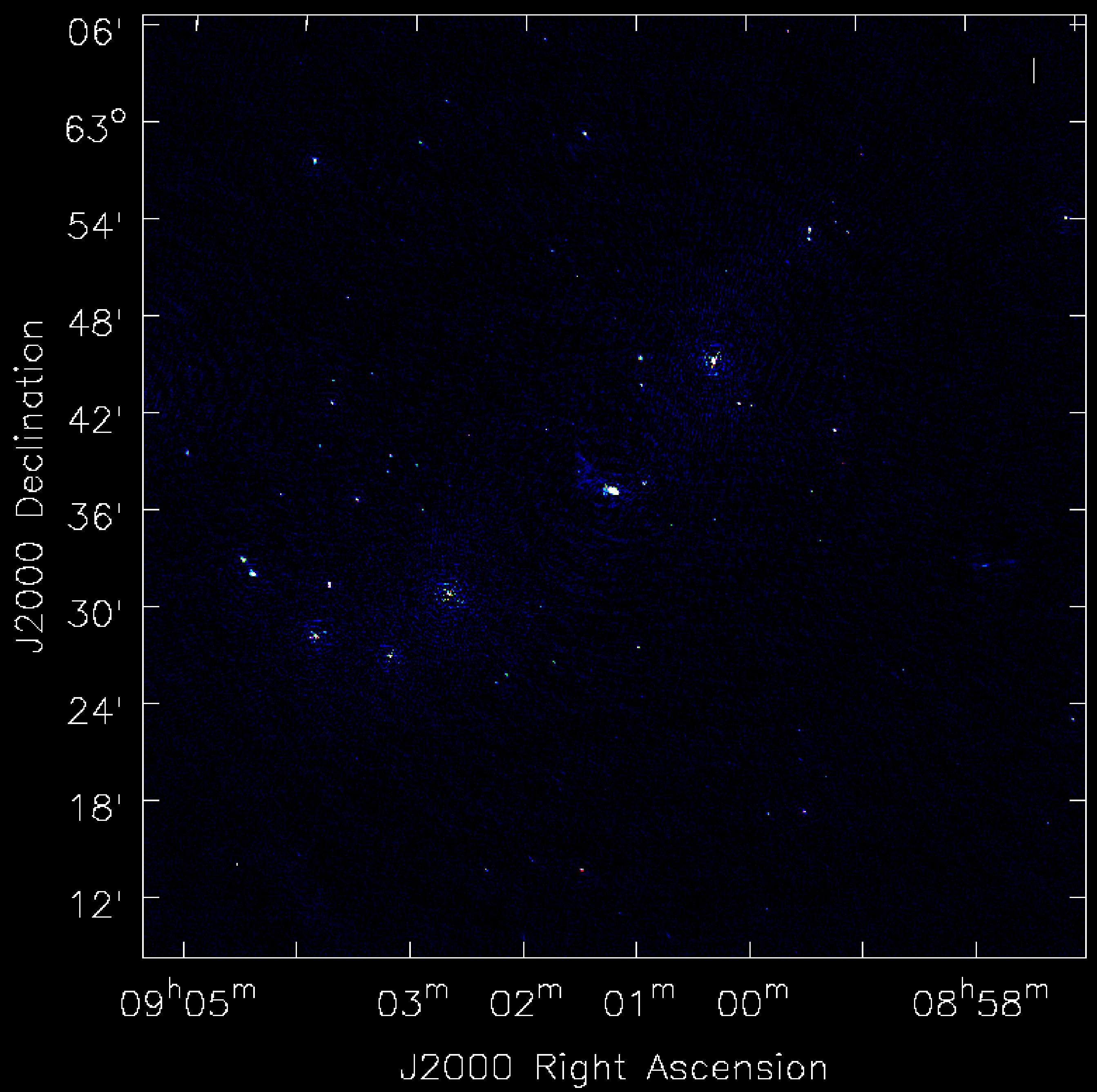

First image after cleaning.

After first round of self-calibration

After second round of self-calibration

GMRT 610 MHz data


06'
63°
54'
48'
42'
36'
30'
24'
18'
12'
J2000 Declination
09h05m
03m
02m
01m
00m
08h58m
J2000 Right Ascension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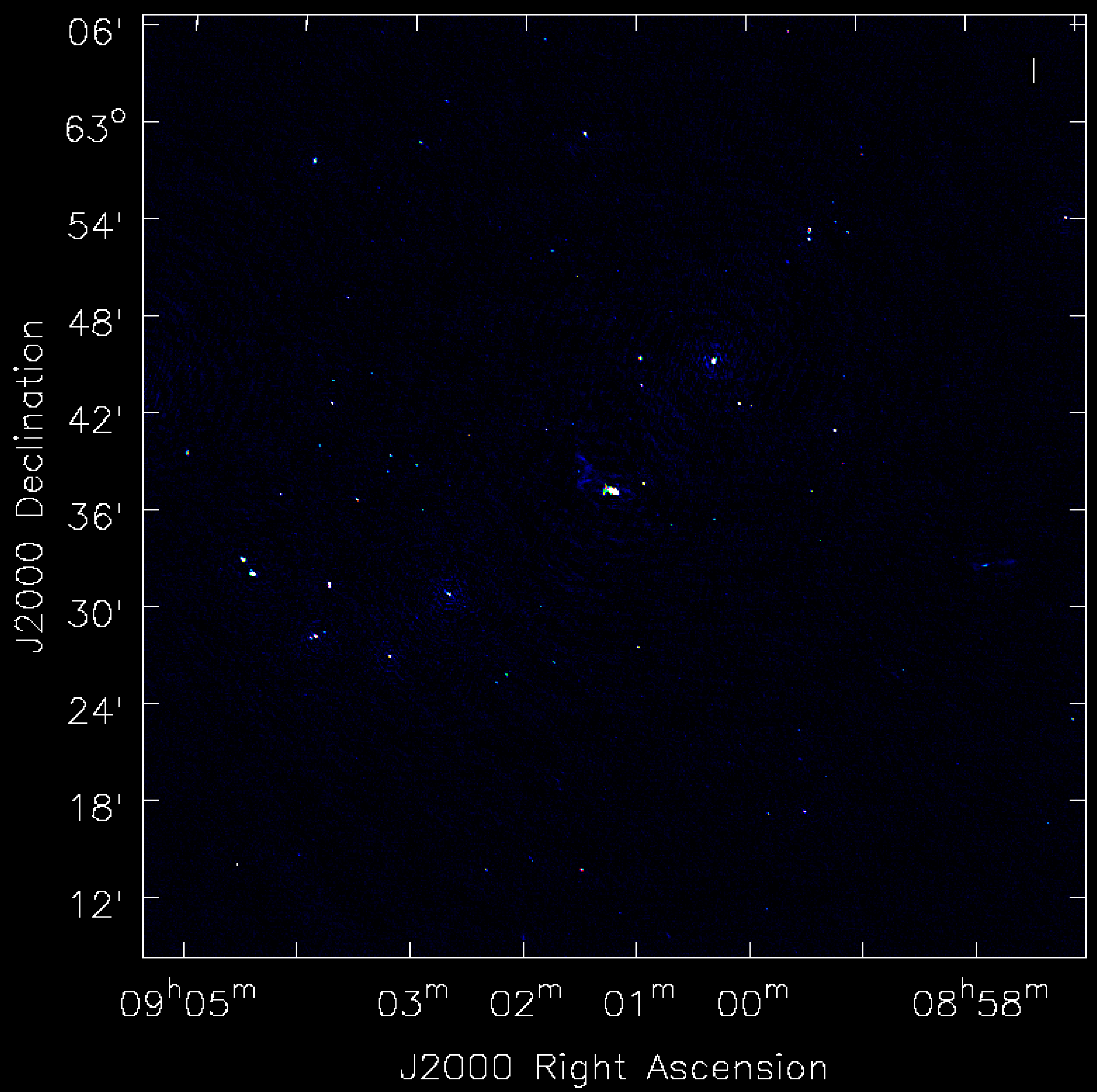
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# Closure quantities and self-calibration

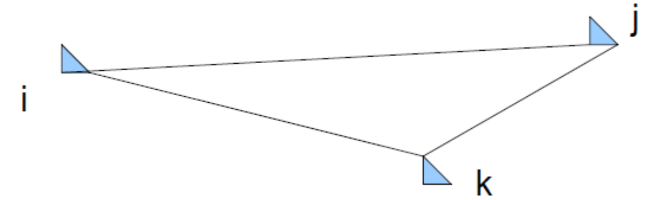
$$\widetilde{V}_{ij}(t) = g_i(t) g_j^*(t) V_{ij}(t) + \epsilon_{ij}(t)$$

Observed True

$$\theta_i(t) = \arg g_i(t)$$

$$\widetilde{\phi}_{ij}(t) = \phi_{ij}(t) + \theta_i(t) - \theta_j(t) + \text{noise term}$$

Considering a loop of three antennas, the observed "closure phase" is,




$$\begin{aligned} \widetilde{C}_{ijk}(t) &= \widetilde{\phi}_{ij}(t) + \widetilde{\phi}_{jk}(t) + \widetilde{\phi}_{ki}(t) \\ &= \phi_{ij}(t) + \phi_{jk}(t) + \phi_{ki}(t) + \text{noise term} \\ &= C_{ijk}(t) + \text{noise term} \end{aligned}$$

Notice that this is independent of the individual errors, thus can be used.



Jennison 1953, 1958

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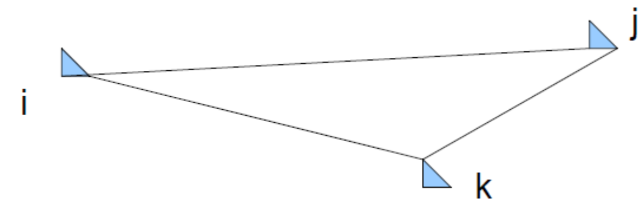


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For an array of N elements, there are ½ N(N-1) – (N-1) independent closure phases - ?



Jennison 1953, 1958

A closure amplitude can be formed for a loop of 4 antennas:

$$\Gamma_{ijkl}(t) = \frac{|\widetilde{V}_{ij}(t)|\,|\widetilde{V}_{kl}(t)|}{|\widetilde{V}_{ik}(t)|\,|\widetilde{V}_{jl}(t)|}$$

Used in self-calibration by Readhead and Wilkinson 1978.

1. Make a model image.
2. for all independent closure phases, use the model to provide estimates of the true phases on two baselines and derive the phase on the other baseline in the loop from the observed phase.
3. form a new model using CLEAN from the observed visibility amplitudes and the predicted phases.
4. Repeat 2 until the model is satisfactory.

Cotton 1979 revised this scheme (least squares technique).

# Sensitivity

Sensitivity is a measure of the weakest source of emission that can be detected.

In radio astronomy power is written in terms of an equivalent temperature, T, of a matched termination on the input of the receiver. Using Rayleigh-Jeans approximation to the Plancks' blackbody radiation, the power is given by:

$$P = k_{\mathrm{B}}\, T\, \Delta\nu$$

$k_B = 1.380 \times 10^{-23}$ J $K^{-1}$

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The power entering the feed is amplified by g in voltage and thus in power,

$$P_\mathrm{a} = g^2\, k_\mathrm{B}\, T_\mathrm{a}\, \Delta\nu$$

From the source

$$P_\mathrm{N} = g^2\, k_\mathrm{B}\, T_\mathrm{sys}\, \Delta\nu$$

From the system

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From the system

$T_a$ is the antenna temperature and
$T_{sys}$ is the system temperature: includes receiver noise, feed losses, spillover, atmospheric emission, galactic background and cosmic background.

# Sensitivity

$$P_a = g^2\, k_B\, T_a\, \Delta\nu$$

The power from the source can be related to the flux density S, the area of the antenna A, the antenna efficiency $\eta_a$ as

$$P_a = \frac{1}{2}\, g^2\, \eta_a\, A\, S\, \Delta\nu$$
$$= g^2\, k_B\, K\, S\, \Delta\nu$$

Radiation received only from one channel; half for an unpolarized source.

Where $K = (\eta_a\, A)\, /\, (2\, k_B)$

"gain" of the antenna in Kelvin of antenna temperature per Jy of flux density

*K* is a measure of antenna performance.

System temperature is often expressed System equivalent flux density (SEFD) : flux density of a source that would deliver the same amount of power

$$SEFD = \frac{T_{sys}}{K} \quad \text{Jy}$$

# SEFD

For 19 antennas in VLBI array at 5 GHz

Taylor et al 1994

**Table 9–1.** Empirical *SEFD*s from Taylor et al. (1994)

| Antenna Location | Diameter (m) | *SEFD* (Jy) |
|---|---|---|
| NRAL, Cambridge, UK | 32 | 140 |
| NRAL, Jodrell Bank, UK | 26 | 366 |
| MPIfR, Effelsberg, Germany | 100 | 39 |
| OSO, Onsala, Sweden | 26 | 757 |
| NFRA, WSRT, Netherlands | 5×25 | 133 |
| IRA, Medicina, Italy | 32 | 225 |
| IRA, Noto, Italy | 32 | 221 |
| Haystack, Westford, MA, USA | 36 | 606 |
| NRAO, Green Bank, WV, USA | 43 | 126 |
| NRAO, VLA, NM, USA | 25 | 319 |
| NRAO, Saint Croix, VI, USA | 25 | 255 |
| NRAO, Hancock, NH, USA | 25 | 259 |
| NRAO, North Liberty, IA, USA | 25 | 300 |
| NRAO, Fort Davis, TX, USA | 25 | 308 |
| NRAO, Los Alamos, NM, USA | 25 | 270 |
| NRAO, Pie Town, NM, USA | 25 | 280 |
| NRAO, Kitt Peak, AZ, USA | 25 | 308 |
| NRAO, Owens Valley, CA, USA | 25 | 249 |
| NRAO, Brewster, WA, USA | 25 | 281 |

# Sensitivity of an interferometer

Consider an interferometer with a single real output which is the product of the voltages from the two elements. The voltage from antenna *i* before sampling is the sum of source voltage and noise voltage. The power from antenna *i* is given by, (factor a includes gain) the expectation value of the square of the voltage:

$$\begin{aligned}\langle P_{\rm i}\rangle &= a_{\rm i}\left\langle (s_{\rm i}+n_{\rm i})^2\right\rangle \\ &= a_{\rm i}\left[\langle s_{\rm i}^2\rangle + \langle n_{\rm i}^2\rangle\right] \\ &= g_{\rm i}^2\, k_{\rm B}\,(T_{\rm ai}+T_{\rm sysi})\,\Delta\nu \\ &= g_{\rm i}^2\, k_{\rm B}\,(K_{\rm i} S_{\rm T}+T_{\rm sysi})\,\Delta\nu\end{aligned}$$

→ Cross terms taken to be zero as voltages from source and noise are uncorrelated.

$S_T$ is the total flux density seen by the antenna and we assume it is same for all the antennas.



Crane and Napier 1989, Walker 1995

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$$\begin{aligned} \langle P_{\rm i} \rangle &= a_{\rm i} \left\langle (s_{\rm i} + n_{\rm i})^2 \right\rangle \\ &= a_{\rm i} \left[ \langle s_{\rm i}^2 \rangle + \langle n_{\rm i}^2 \rangle \right] \\ &= g_{\rm i}^2 \, k_{\rm B} \, (T_{\rm ai} + T_{\rm sysi}) \, \Delta\nu \\ &= g_{\rm i}^2 \, k_{\rm B} \, (K_{\rm i} S_{\rm T} + T_{\rm sysi}) \, \Delta\nu \end{aligned}$$

Cross terms taken to be zero as voltages from source and noise are uncorrelated.

$S_T$ is the total flux density seen by the antenna and we assume it is same for all the antennas.

The power after cross multiplication in the correlator can be obtained as (i and j are the antennas):

$$\begin{aligned} \langle P_{\rm ij} \rangle &= \frac{\sqrt{a_{\rm i} a_{\rm j}}}{\eta_{\rm s}} \Big\langle (s_{\rm i} + n_{\rm i})(s_{\rm j} + n_{\rm j}) \Big\rangle \\ &= \frac{\sqrt{a_{\rm i} a_{\rm j}}}{\eta_{\rm s}} \langle s_{\rm i} s_{\rm j} \rangle \\ &= \frac{g_{\rm i} \, g_{\rm j}}{\eta_{\rm s}} \sqrt{K_{\rm i} \, K_{\rm j}} \, k_{\rm B} \, \Delta\nu \, S_{\rm c} \end{aligned}$$

Efficiency factor that accounts for losses in the electronics and digital equipment

Noise from the two antennas is uncorrelated

$S_c$ is correlated flux density



Crane and Napier 1989, Walker 1995

To obtain the signal-noise-ratio (SNR) we look at the RMS fluctuations of the correlator output – consider rms fluctuations of the product of the antenna voltages for each sample.
We use the fact that the square of RMS fluctuation of a Gaussian random variable (product from the correlator) is equal to the expectation value of the variable squared minus the square of the mean.

$$\sigma^2(P_{\rm ij}) = \frac{a_{\rm i}a_{\rm j}}{\eta_{\rm s}^2}\left\langle \left[(s_{\rm i}+n_{\rm i})(s_{\rm j}+n_{\rm j})\right]^2 \right\rangle - \frac{g_{\rm i}^2\, g_{\rm j}^2}{\eta_{\rm s}^2} K_{\rm i}\, K_{\rm j}\, k_{\rm B}^2\, S_{\rm c}^2\, \Delta\nu^2$$

Recall

$$\langle P_{\rm ij}\rangle = \frac{\sqrt{a_{\rm i}a_{\rm j}}}{\eta_{\rm s}}\left\langle (s_{\rm i}+n_{\rm i})(s_{\rm j}+n_{\rm j})\right\rangle = \frac{g_{\rm i}\, g_{\rm j}}{\eta_{\rm s}}\sqrt{K_{\rm i}\, K_{\rm j}}\, k_{\rm B}\, \Delta\nu\, S_{\rm c}$$

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$$\sigma^2(P_{ij}) = \frac{a_i a_j}{\eta_s^2} \left\langle \left[ (s_i + n_i)(s_j + n_j) \right]^2 \right\rangle - \frac{g_i^2\, g_j^2}{\eta_s^2} K_i\, K_j\, k_B^2\, S_c^2\, \Delta\nu^2$$

$$\downarrow$$

$$\frac{a_i a_j}{\eta_s^2} \left[ 2 \Big\langle (s_i + n_i)(s_j + n_j) \Big\rangle^2 + \Big\langle (s_i + n_i)^2 \Big\rangle \Big\langle (s_j + n_j)^2 \Big\rangle \right]$$

A standard relation to expand the expectation value of a product of four variables into combinations of expectation values of products of two variables is used here.

Assuming all the processes involved are Gaussian processes then we can use the properties of these processes that are already known.
If x1, x2, x3 and x4 have a joint Gaussian distribution then,

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle$$

$$\left\langle \left[ (s_i + n_i)(s_j + n_j) \right]^2 \right\rangle$$

$$\rightarrow \left[ 2\left\langle (s_i + n_i)(s_j + n_j) \right\rangle^2 + \left\langle (s_i + n_i)^2 \right\rangle \left\langle (s_j + n_j)^2 \right\rangle \right]$$

See chp 5 of LFRA.

To obtain SNR we look at the RMS fluctuations of the correlator output. We use the fact that the square of RMS fluctuation of a Gaussian random variable (product from the correlator) is equal to  the expectation value of the variable squared minus the square of the mean.

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$$\begin{aligned}
\sigma^2(P_{\rm ij}) &= \frac{a_{\rm i}a_{\rm j}}{\eta_{\rm s}^2}\left[2\left\langle(s_{\rm i}+n_{\rm i})(s_{\rm j}+n_{\rm j})\right\rangle^2 + \left\langle(s_{\rm i}+n_{\rm i})^2\right\rangle\left\langle(s_{\rm j}+n_{\rm j})^2\right\rangle\right] \\
&\quad - \frac{g_{\rm i}^2\, g_{\rm j}^2}{\eta_{\rm s}^2} K_{\rm i}\, K_{\rm j}\, k_{\rm B}^2\, \Delta\nu^2\, S_{\rm c}^2 \\
&= 2\frac{g_{\rm i}^2 g_{\rm j}^2}{\eta_{\rm s}^2} K_{\rm i}\, K_{\rm j}\, (k_{\rm B}\, \Delta\nu\, S_{\rm T})^2 \\
&\quad + \frac{g_{\rm i}^2\, g_{\rm j}^2}{\eta_{\rm s}^2} (k_{\rm B}\, \Delta\nu)^2 (K_{\rm i}\, S_{\rm T} + T_{\rm sysi})(K_{\rm j}\, S_{\rm T} + T_{\rm sysj}) \\
&\quad - \frac{g_{\rm i}^2\, g_{\rm j}^2}{\eta_{\rm s}^2} K_{\rm i}\, K_{\rm j}\, k_{\rm B}^2\, \Delta\nu^2\, S_{\rm c}^2
\end{aligned}$$

$$\begin{aligned}
\langle P_{\rm i}\rangle &= a_{\rm i}\left\langle(s_{\rm i}+n_{\rm i})^2\right\rangle \\
&= g_{\rm i}^2\, k_{\rm B}\, (K_{\rm i} S_{\rm T} + T_{\rm sysi})\, \Delta\nu
\end{aligned}$$

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$$\begin{aligned}\sigma^2(P_{\mathrm{ij}}) &= \frac{a_{\mathrm{i}}a_{\mathrm{j}}}{\eta_{\mathrm{s}}^2}\left[2\left\langle(s_{\mathrm{i}}+n_{\mathrm{i}})(s_{\mathrm{j}}+n_{\mathrm{j}})\right\rangle^2 + \left\langle(s_{\mathrm{i}}+n_{\mathrm{i}})^2\right\rangle\left\langle(s_{\mathrm{j}}+n_{\mathrm{j}})^2\right\rangle\right] \\ &\quad - \frac{g_{\mathrm{i}}^2\, g_{\mathrm{j}}^2}{\eta_{\mathrm{s}}^2} K_{\mathrm{i}}\, K_{\mathrm{j}}\, k_{\mathrm{B}}^2\, \Delta\nu^2\, S_{\mathrm{c}}^2 \\ &= 2\frac{g_{\mathrm{i}}^2 g_{\mathrm{j}}^2}{\eta_{\mathrm{s}}^2} K_{\mathrm{i}}\, K_{\mathrm{j}}\, (k_{\mathrm{B}}\, \Delta\nu\, S_{\mathrm{c}})^2 \\ &\quad + \frac{g_{\mathrm{i}}^2\, g_{\mathrm{j}}^2}{\eta_{\mathrm{s}}^2}(k_{\mathrm{B}}\, \Delta\nu)^2 (K_{\mathrm{i}}\, S_{\mathrm{T}} + T_{\mathrm{sysi}})(K_{\mathrm{j}}\, S_{\mathrm{T}} + T_{\mathrm{sysj}}) \\ &\quad - \frac{g_{\mathrm{i}}^2\, g_{\mathrm{j}}^2}{\eta_{\mathrm{s}}^2} K_{\mathrm{i}}\, K_{\mathrm{j}}\, k_{\mathrm{B}}^2\, \Delta\nu^2\, S_{\mathrm{c}}^2\end{aligned}$$

$$\begin{aligned}\langle P_{\mathrm{ij}}\rangle &= \frac{\sqrt{a_{\mathrm{i}}a_{\mathrm{j}}}}{\eta_{\mathrm{s}}}\left\langle (s_{\mathrm{i}}+n_{\mathrm{i}})(s_{\mathrm{j}}+n_{\mathrm{j}})\right\rangle \\ &= \frac{g_{\mathrm{i}}\, g_{\mathrm{j}}}{\eta_{\mathrm{s}}}\sqrt{K_{\mathrm{i}}\, K_{\mathrm{j}}}\, k_{\mathrm{B}}\, \Delta\nu\, S_{\mathrm{c}}\end{aligned}$$

$$\sigma^2(P_{ij}) = k_B^2\,\Delta\nu^2\,\frac{g_i^2\,g_j^2}{\eta_s^2}\Big(K_i\,K_j\,S_c^2 + K_i\,K_j\,S_T^2 + K_i\,S_T\,T_{sysi} + K_j\,S_T\,T_{sysj} + T_{sysi}\,T_{sysj}\Big)$$

To obtain the noise level in units of flux density, divide by $g_i\,g_j\sqrt{K_i\,K_j}\,k_B\,\Delta\nu$

Source flux density to cross correlated power

and $\sqrt{2\,\Delta\nu\,\tau_{acc}}$ the standard deviation of the mean

$$\Delta S_{ij} = \frac{1}{\eta_s\sqrt{2\,\Delta\nu\,\tau_{acc}}}\sqrt{S_c^2 + S_T^2 + S_T\left(\frac{T_{sysi}}{K_i} + \frac{T_{sysj}}{K_j}\right) + \frac{T_{sysi}\,T_{sysj}}{K_i\,K_j}}$$



Crane and Napier 1989

$$\Delta S_{\rm ij} = \frac{1}{\eta_{\rm s}\sqrt{2\,\Delta\nu\,\tau_{\rm acc}}}\sqrt{S_{\rm c}^2 + S_{\rm T}^2 + S_{\rm T}\left(\frac{T_{\rm sysi}}{K_{\rm i}} + \frac{T_{\rm sysj}}{K_{\rm j}}\right) + \frac{T_{\rm sysi}\,T_{\rm sysj}}{K_{\rm i}\,K_{\rm j}}}$$

A square bandpass is assumed here.

For a non flat bandpass: $\int_0^\infty g_{\rm i}(\nu)\,g_{\rm j}(\nu)\,d\nu$

$$\Delta S_{\rm ij} = \frac{\sqrt{\int_0^\infty g_{\rm i}^2(\nu)\,g_{\rm j}^2(\nu)\left[S_{\rm c}^2 + S_{\rm T}^2 + S_{\rm T}\left(\frac{T_{\rm sysi}}{K_{\rm i}} + \frac{T_{\rm sysj}}{K_{\rm j}}\right) + \frac{T_{\rm sysi}\,T_{\rm sysj}}{K_{\rm i}\,K_{\rm j}}\right]d\nu}}{\eta_{\rm s}\sqrt{2\,\tau_{\rm acc}\int_0^\infty g_{\rm i}(\nu)\,g_{\rm j}(\nu)\,d\nu}}$$

The noise of the correlated signal, $S_c$ and of the source power as it adds to the total powers at each antenna, $S_T$, both contribute to the total noise of the correlated output.

# Special cases assuming flat bandpass

$$\Delta S_{\mathrm{ij}} = \frac{1}{\eta_{\mathrm{s}}\sqrt{2\,\Delta\nu\,\tau_{\mathrm{acc}}}}\sqrt{S_{\mathrm{c}}^2 + S_{\mathrm{T}}^2 + S_{\mathrm{T}}\left(\frac{T_{\mathrm{sysi}}}{K_{\mathrm{i}}} + \frac{T_{\mathrm{sysj}}}{K_{\mathrm{j}}}\right) + \frac{T_{\mathrm{sysi}}\,T_{\mathrm{sysj}}}{K_{\mathrm{i}}\,K_{\mathrm{j}}}}$$

Case of strong source:

$$S_{\mathrm{T}} \gg S_{\mathrm{c}}.$$

$$\Delta S_{\mathrm{ij}} = \frac{S_{\mathrm{T}}}{\eta_{\mathrm{s}}\sqrt{2\,\Delta\nu\,\tau_{\mathrm{acc}}}}$$

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Typically source is weak and thus ignoring terms involving flux density S:

$$\Delta S_{ij} = \frac{1}{\eta_s} \sqrt{\frac{T_{sysi}\,T_{sysj}}{2\,\Delta\nu\,\tau_{acc}\,K_i\,K_j}}$$

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In terms of SEFD:

$$SEFD = \frac{T_{sys}}{K}$$

$$\Delta S_{ij} = \frac{1}{\eta_s} \sqrt{\frac{SEFD_i\,SEFD_j}{2\,\Delta\nu\,\tau_{acc}}}$$

If SEFD same for two antennas:

$$\Delta S_{ij} = \frac{1}{\eta_s} \frac{SEFD}{\sqrt{2\,\Delta\nu\,\tau_{acc}}}$$

# Sensitivity of an image

$$I_m(l,m) = C \sum_{k=1}^{2L} T_k \, W_k \, w_k \, V_k \, e^{2\pi i(u_k l + v_k m)}$$

Normalisation constant

k=0 term represents zero-spacing – not available

2L: visibility function is Hermitian – the conjugate points will also be known

Taper

Weights: Natural, uniform, etc

Weight based on S/N of individual point: important for VLBI but not for arrays with identical elements

*Each image pixel is a linear combination of each measured data point.*

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Consider only the centre of the image:

$$I_m(0,0) = 2\,C \sum_{k=1}^{L} T_k \, W_k \, w_k \, S_{Rk}$$

$$\Delta S_k$$

Image has only real values $S_R$

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Image has only real values $S_R$

Error in this image point is due to error $\Delta S_k$ ; thus variance in I is just the sum of variances in the Fourier components:

$$\Delta I_m = 2\,C \sqrt{\sum_{k=1}^{L} T_k^2 \, W_k^2 \, w_k^2 \, \Delta S_k^2}$$

C is set to: $1 \,/\, (2 \sum_{k=1}^{L} T_k \, W_k \, w_k)$

To obtain the result in terms of flux density per beam area

$$\Delta I_m = 2\,C\,\sqrt{\sum_{k=1}^{L} T_k^2\,W_k^2\,w_k^2\,\Delta S_k^2}$$

For no taper, natural weights, and all points of equal SNR

$w_k$= w,

$T_k$ =1; $W_k$ =1

$$\begin{aligned}
\Delta I_m &= 2\,C\,\Delta S\,w\sqrt{\sum_{k=1}^{L} 1} \\
&= \frac{2\,\Delta S\,w\sqrt{\sum_{k=1}^{L} 1}}{2\,w\sum_{k=1}^{L} 1} \\
&= \Delta S\sqrt{L}/L \\
&= \Delta S/\sqrt{L}
\end{aligned}$$

C= $1\,/\,(2\sum_{k=1}^{L} T_k\,W_k\,w_k)$

$$\Delta I_{\mathrm{m}} = 2\,C\,\sqrt{\sum_{\mathrm{k}=1}^{L} T_{\mathrm{k}}^2\,W_{\mathrm{k}}^2\,w_{\mathrm{k}}^2\,\Delta S_{\mathrm{k}}^2}$$

For no taper, natural weights, and all points of weight w,

$$\begin{aligned}\Delta I_{\mathrm{m}} &= 2\,C\,\Delta S\,w\sqrt{\sum_{\mathrm{k}=1}^{L} 1} \\ &= \frac{2\,\Delta S\,w\sqrt{\sum_{\mathrm{k}=1}^{L} 1}}{2\,w\sum_{\mathrm{k}=1}^{L} 1} \\ &= \Delta S\sqrt{L}/L \\ &= \Delta S/\sqrt{L}\end{aligned}$$

N= number of antennas

$$L = \frac{1}{2}\,N\,(N-1)\,(t_{\mathrm{int}}/\tau_{\mathrm{acc}})$$

$$\Delta I_{\mathrm{m}} = \frac{1}{\eta_{\mathrm{s}}}\frac{SEFD}{\sqrt{N\,(N-1)\,\Delta\nu\,t_{\mathrm{int}}}}$$

Sensitivity of a *single* polarization image formed from a homogenous array of N identical antennas:

$$\Delta I_{\rm m} = \frac{1}{\eta_{\rm s}} \frac{SEFD}{\sqrt{N\,(N-1)\,\Delta\nu\,t_{\rm int}}}$$

For full Stokes data:

$$\Delta I = \Delta Q = \Delta U = \Delta V = \frac{\Delta I_{\rm m}}{\sqrt{2}}$$

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For full Stokes data:

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Noise in linear polarized flux density follows Rayleigh statistics and the position angle will follow uniform statistics.

$$P = \sqrt{Q^2 + U^2} \qquad \chi = \frac{1}{2}\tan^{-1}(U/Q)$$

See Interferometry and Synthesis in RA for more details.

# Sensitivity

For a single antenna:

$$\sigma_s = \frac{2kT_{\rm sys}}{A_{\rm eff}(\Delta\nu_{\rm RF}\tau)^{1/2}}$$

For an interferometer:

$$\sigma_s = \frac{2^{1/2}kT_{\rm sys}}{A_{\rm eff}(\Delta\nu_{\rm RF}\tau)^{1/2}}$$

*The sensitivity of an interferometer is square root 2 times better than that of each antenna but the same factor worse than a single antenna with an area of two antennas.*

For an interferometer with N elements:

$$\sigma_s = \frac{2kT_{\rm sys}}{A_{\rm eff}\left[N(N-1)\Delta\nu_{\rm RF}\tau\right]^{1/2}}$$

In the limit of large N, N(N-1) → $N^2$ and the point source sensitivity of an interferometer approaches that of a single antenna whose area equals the total effective area $NA_{eff}$ of the interferometer.

In practice, interferometers are slightly less sensitive due to the effects of sampling and digitizing.