Electrodynamics 2, Special relativity worksheet
Oct 23, 2007

1) Lorentz transformation between S & S', general direction $\hat{\beta}$, velocity of S' w.r.t $S = c\vec{\beta}$, $\vec{r} = (x, y, z)$ & $\vec{r}' = (x', y', z')$ are both decomposed into components $\parallel$e & $\perp$r to $\hat{\beta}$, i.e. $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$; $\vec{r}_{\parallel} = (\vec{r}\cdot\hat{\beta})\hat{\beta}$, $\vec{r}_{\perp} = \vec{r} - (\vec{r}\cdot\hat{\beta})\hat{\beta}$, similarly for $\vec{r}'$

Then we have $\vec{r}'_{\perp} = \vec{r}_{\perp}$;

$$\vec{r}'_{\parallel} = \gamma(\vec{r}_{\parallel} - \vec{\beta}(ct))\ ;\ ct' = \gamma(ct - \vec{\beta}\cdot\vec{r})$$

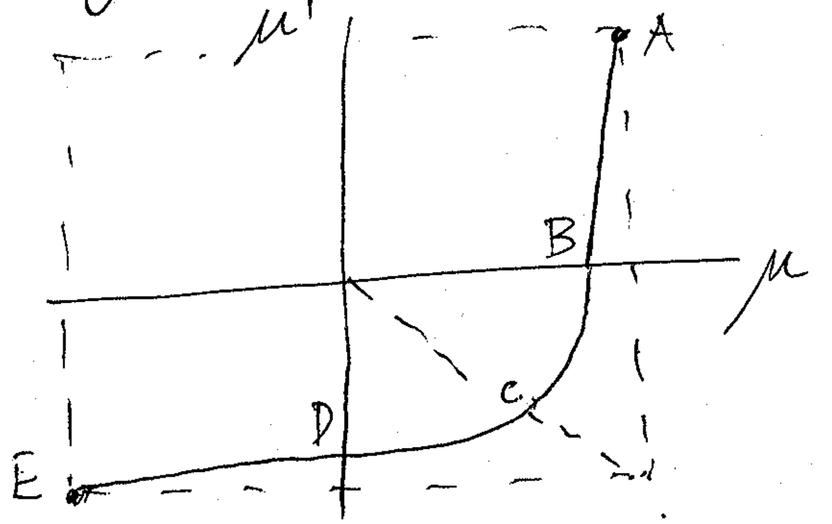
2) Other 4 vectors transforming similarly to $(ct, \vec{r})$ are $(E, c\vec{p})$; $(\vec{J}, c\rho)$ and $(\omega, c\vec{k})$. The last is a four vector for any wave, but for EM waves is a null 4 vector because $\omega = c|\vec{k}|$

Its transformation law therefore splits into the relativistic aberration & Doppler effect formulae. Notation: $\mu = \hat{k}\cdot\hat{\beta} = \cos\theta$, θ = angle made by light ray/wave normal/photon to the relative velocity direction, similarly, μ'

Doppler effect: $\frac{\omega'}{\omega} = \gamma(1-\beta\mu)$; sometimes called the 'Doppler factor' δ

$\mu' = (\mu-\beta) / (1-\beta\mu)$. The reverse formulae can be obtained either by algebra or by changing β to $-\beta$

3) Non relativistically ($\beta \ll 1$), we are used to $\frac{\Delta\omega}{\omega} = -\beta\mu$, and $\Delta\theta = +\beta\sin\theta$, ie Doppler effect is given by radial velocity/c & aberration by transverse velocity/c. However, its worth getting used to the fully relativistic case



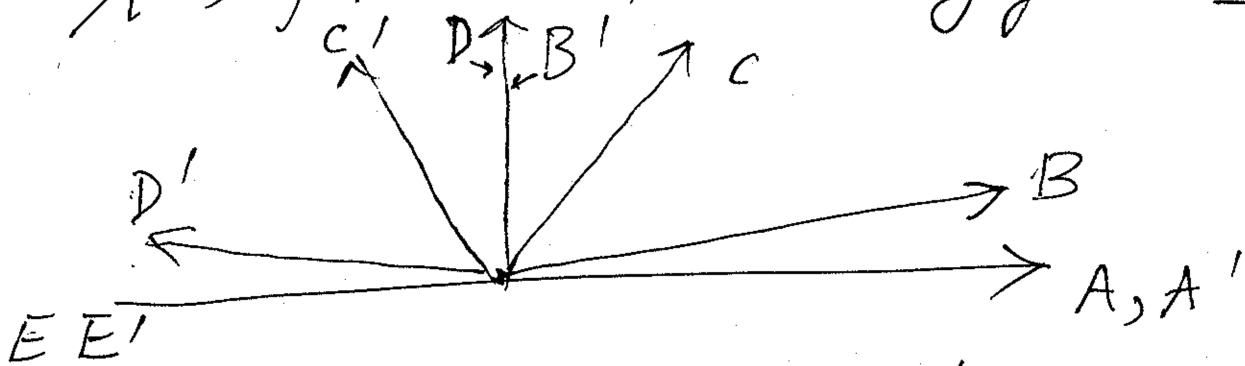
On the plot of μ' vs. μ, five points ABCDE are shown.

A & E simply tell us that light along the forward (A) or backward (E) direction in one frame remains so in the other - forward/backward w.r.t the relative velocity $\vec{\beta}$.

The point B, $\mu' = 0$, $\mu = \beta$, tells us that a beam very close to the forward direction can get moved to 90° in the primed frame. D tells us that a beam at 90° gets moved very close to backwards.

Quantitatively, since $\beta \approx 1 - \frac{1}{2\gamma^2}$ for $\gamma \gg 1$ & $\cos\theta = 1 - \frac{\theta^2}{2}$ for $\theta \ll 1$, we have B corresponding to $\theta = \frac{1}{\gamma}$ (& similarly D is $\theta' = -\pi + 1/\gamma$)

Finally, C corresponds to the case $\mu' = -\mu$, for which, interestingly $\delta = 1$

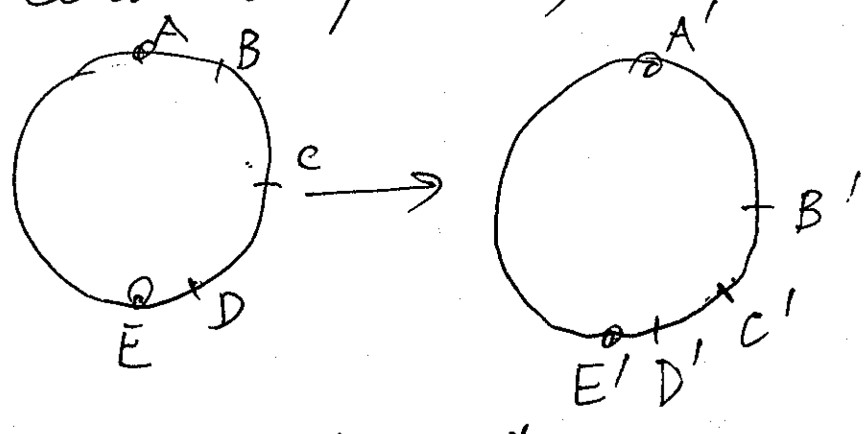


One can check that the angles corresponding to C & C' are, approximately $\theta \approx \sqrt{\frac{2}{\gamma}}$ & $\theta' = -\sqrt{\frac{2}{\gamma}}$ for $\gamma \gg 1$

4) Taking $\hat{\beta}$ as the polar axis, the angle φ remains the same, this is best seen by using $\tan\varphi' = \frac{k_y'}{k_x'} = \frac{k_y}{k_x} = \tan\varphi$. The element of solid angle $d\Omega' = \sin\theta' d\theta' d\varphi' = d\mu' d\varphi'$ is related to $d\Omega$ by the inverse square of the Doppler factor – pl. check)

$$d\Omega' = \frac{d\Omega}{\gamma^2(1-\beta\mu)^2}.$$

We can think of aberration as mapping the sphere of arrival directions of light (celestial sphere!) onto itself.



Solid angle is not preserved, but (check that!) the ratio of sizes in the θ & φ directions is preserved – aberration is a conformal transformation. And follow Penrose to show a finite circle maps to a circle.

5) The law of transformation of the field components $\vec{E}$ & $\vec{B}$ can be worked out using that for a general 2nd rank tensor, e.g. $E_x' = F_{01}'$; etc. The result, in 3+1 form, looks strangely like that for 4 vectors, for reasons explained in Landau Lifshitz v.2

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel}, \quad \vec{B}_{\parallel}' = \vec{B}_{\parallel}, \quad \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{\beta}\times B_{\perp})$$

$B_{\perp}' = \gamma(B_{\perp} - \vec{\beta}\times\vec{E}_{\perp})$. Even more concisely,

$$(\vec{E} + i\vec{B})_{\parallel}' = (\vec{E} + i\vec{B}); \quad \text{and}$$

$$(E_{\perp} + iB_{\perp})' = \gamma\left((\vec{E}_{\perp} + i\vec{B}_{\perp}) - i\beta(\vec{E}_{\perp} + i\vec{B}_{\perp})\right)$$

The utility of $\vec{E} + i\vec{B}$ is not an accident, nor is the resemblance to 4 vectors ..... Multiplying $(\vec{E} + iB)$ by i is like replacing $\vec{E}$ by $-\vec{B}$ & $\vec{B}$ by $\vec{E}$, which is the electric-magnetic duality of the vacuum (no sources) Maxwell equations.

(6) One can use the field transformations, (6)
plus ~~the fact that~~ $|B| = |E|$,
to directly check that $|\vec{E}|$ transforms
according to the Doppler factor δ.
i.e $|E|' = \gamma(1-\beta\mu)\,|\vec{E}|$, or $\frac{|E|}{\omega}$ is
Lorentz invariant. This tells us that
the Poynting flux $|\vec{S}| = \frac{c\,|E|^2}{4\pi}$ acquires
a δ^2. This δ^2 can be understood
by noting that (a) the length of a
wave packet transforms in the same way
as the wavelength, i.e a $1/\delta$
(b) The energy of each photon picks up δ.
(c) The area of the beam is invariant
(trivial if $\hat{k} \parallel \hat{\beta}$ ie $\mu = 1$, but true
in general).
So $|\vec{S}|$ which is energy/time/unit area
gets a δ^2. If we make it flux
density, i.e per unit freq interval,
it becomes a δ. And if we look
at specific intensity I (add per unit
solid angle) than it is δ^3.

This implies that I/ω^3 is Lorentz covariant.

(7) It is useful to know how acceleration transforms from the rest frame S (particle is at the origin with zero velocity at $t = 0$) to a general frame S'. For small times, the world line is either $y = \frac{1}{2} a_{\perp r} t^2$ ($\vec{a}_{rest} \perp \hat{\beta}$) or $x = \frac{1}{2} a_{\parallel r} t^2$.

One can now replace by y', x', t' & get the acceleration in the lab frame. The result is

$a'_\perp = a_\perp / \gamma^2$ (two time dilation factors)

and $a'_\parallel = a_\parallel / \gamma^3$. This is not simple time dilation because x' is not equal to x.

(8) In its instantaneous rest frame, the charge radiates ΔE, but $\Delta \vec{p} = 0$. Its displacement four vector is $(\Delta t, \vec{0})$. The proportionality is preserved in S', hence $\Delta E'/\Delta t' = \Delta E / \Delta t$