Electrodynamics and Radiative Processes I

Lecture 9 – Synchrotron Radiation I

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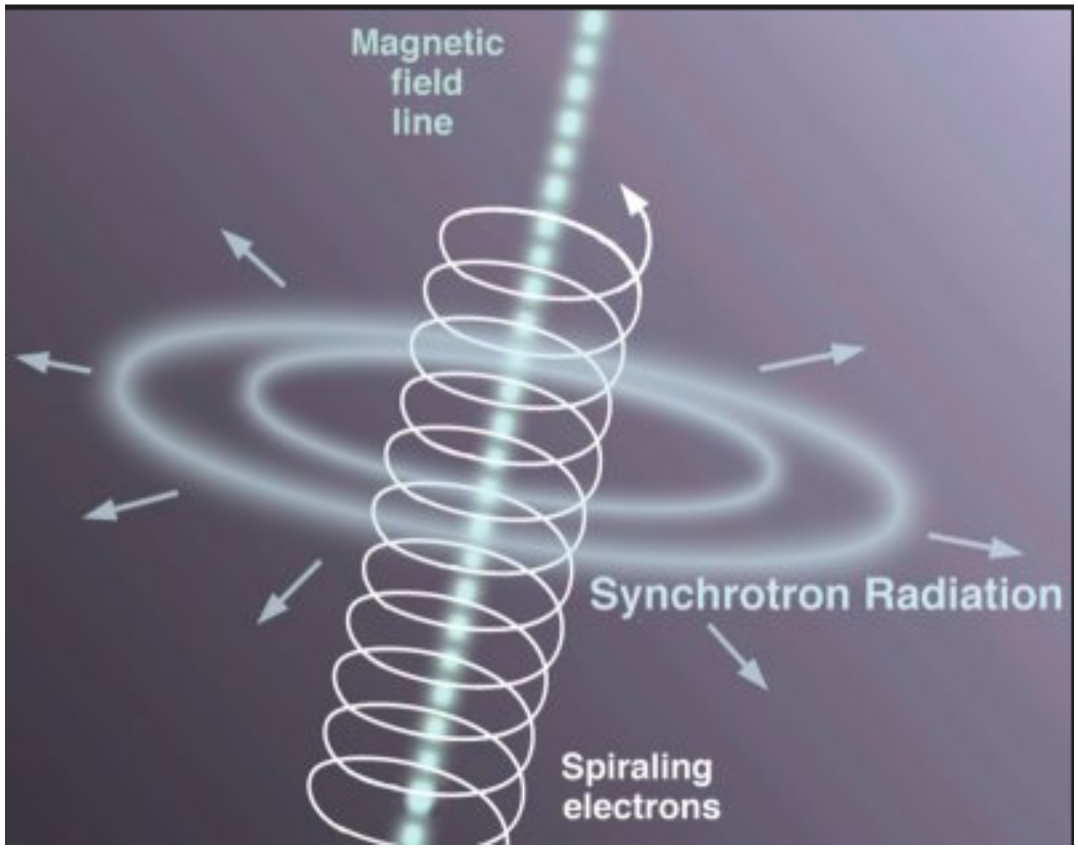
Synchrotron Radiation

Synchrotron Radiation is emitted from particles accelerated by a magnetic field and moving relativistically



To understand synchrotron radiation let's first begin with the non-relativistic motion of a charge accelerated by a magnetic field : Cyclotron radiation

Cyclotron radiation

Particles accelerated by the magnetic field will radiate

For nonrelativistic velocities nature of radiation is called **Cyclotron radiation**

Frequency of emission is frequency of gyration in the magnetic field

Cyclotron radiation

Accelerated charged particle will radiate according to the Larmor formula (covered in Lecture 4)

$$P = \frac{2q^2\dot{u}^2}{3c^3}$$

It does not matter if the acceleration is given by electric field, gravity or magnetic field

Larmor's Formula

Total power radiated by a non-relativistic point charge as it accelerates

$$\mathbf{E}_{\rm rad}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

For β<<1

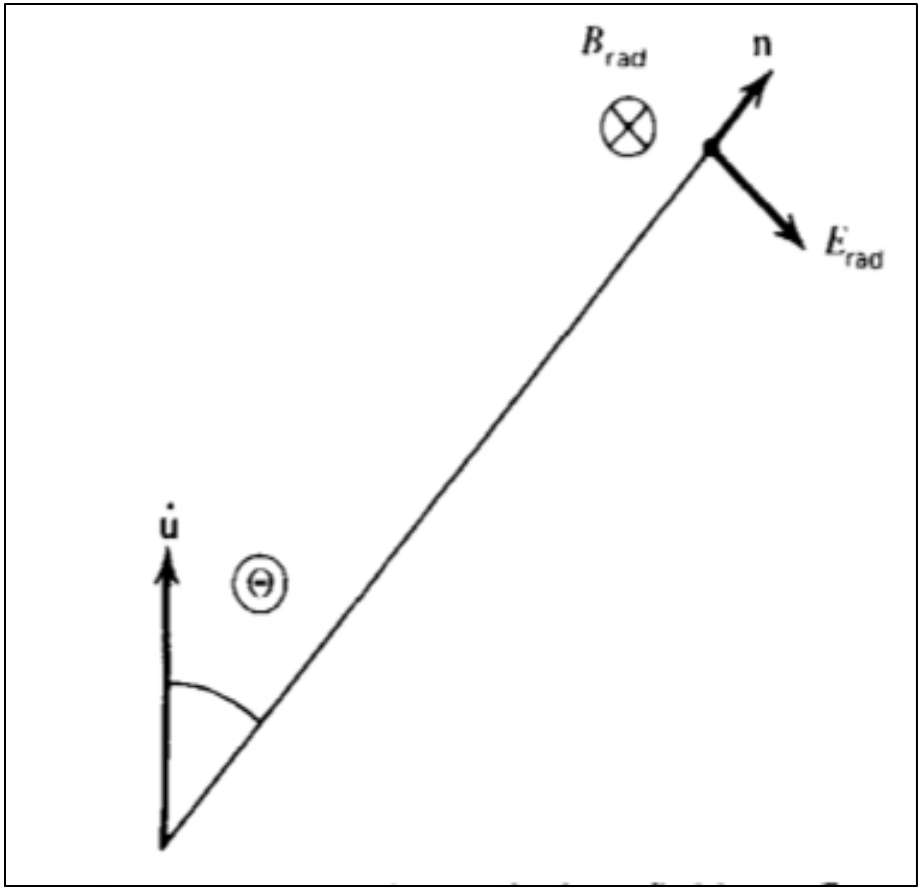
$$\mathbf{E}_{\rm rad}=\left[(q/Rc^2)\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]$$

$$\mathbf{B}_{\rm rad}=\left[\mathbf{n}\times\mathbf{E}_{\rm rad}\right]$$

$$|\mathbf{E}_{\rm rad}|=|\mathbf{B}_{\rm rad}|=\frac{q\dot{u}}{Rc^2}\sin\Theta$$

Poynting Vector

$$S=\frac{c}{4\pi}E_{\rm rad}^2=\frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta$$



Outward flow of energy along **n**

Larmor's Formula

$$S = \frac{c}{4\pi} E_{\text{rad}}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2\Theta$$

Power radiated per unit solid angle per unit time

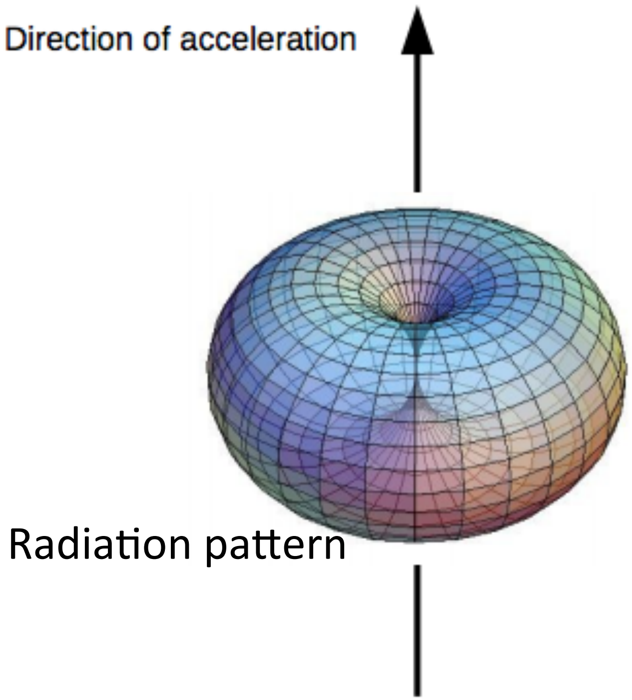
$$\frac{dW}{dt\,d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\Theta. \quad \Rightarrow \quad P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int \sin^2\Theta \, d\Omega$$

$$\boxed{P = \frac{2 q^2 \dot{u}^2}{3 c^3}}$$

$\Downarrow$

Larmor's Formula for emission
from a single accelerated charge q

Cyclotron radiation



Energy per unit time per unit solid angle

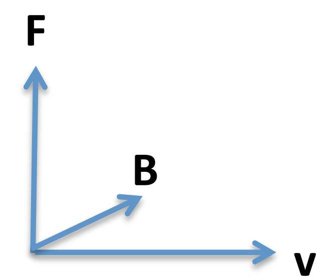
$$\frac{dW}{dt\,d\Omega} \approx \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2\Theta$$

The radiation pattern is a torus with Sin^2 dependence of angle of radiation

Cyclotron radiation

Let us take a charge (say q) and put it in uniform magnetic field B

Force $F = q\,\mathbf{v} \times \mathbf{B} = q\,v\,B$ (If B is orthogonal to v)



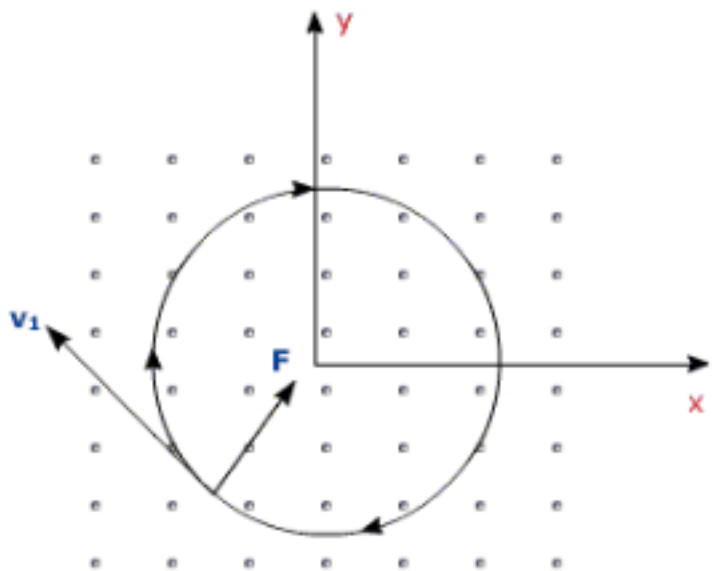
Force $F = q\,\mathbf{v} \times \mathbf{B} = q\,v\,B = mv^2/r_L$ = Centripetal force

Larmor Radius /Gyro Radius $\boxed{r_L = mv/qB}$

Force $F = mv^2/r_L = m\,\omega_L r_L$

Cyclotron frequency $\boxed{\omega_L = qB/m}$

Cyclotron radiation



From angular frequency we can find period of rotation of the charge

$$T=2\pi/\omega_L=2\pi m/qB$$

Period of the particle is not dependent on the size of orbit

Period of the particle is constant if B is constant

Cyclotron radiation

From angular frequency we can find period of rotation of the charge

$$T=2\pi/\omega_L=2\pi m/qB$$

The charge that is rotating will emit at a single specific frequency

$$\nu_L=\omega_L/2\pi=qB/2\pi m =2.8 \text{ MHz per Gauss}$$

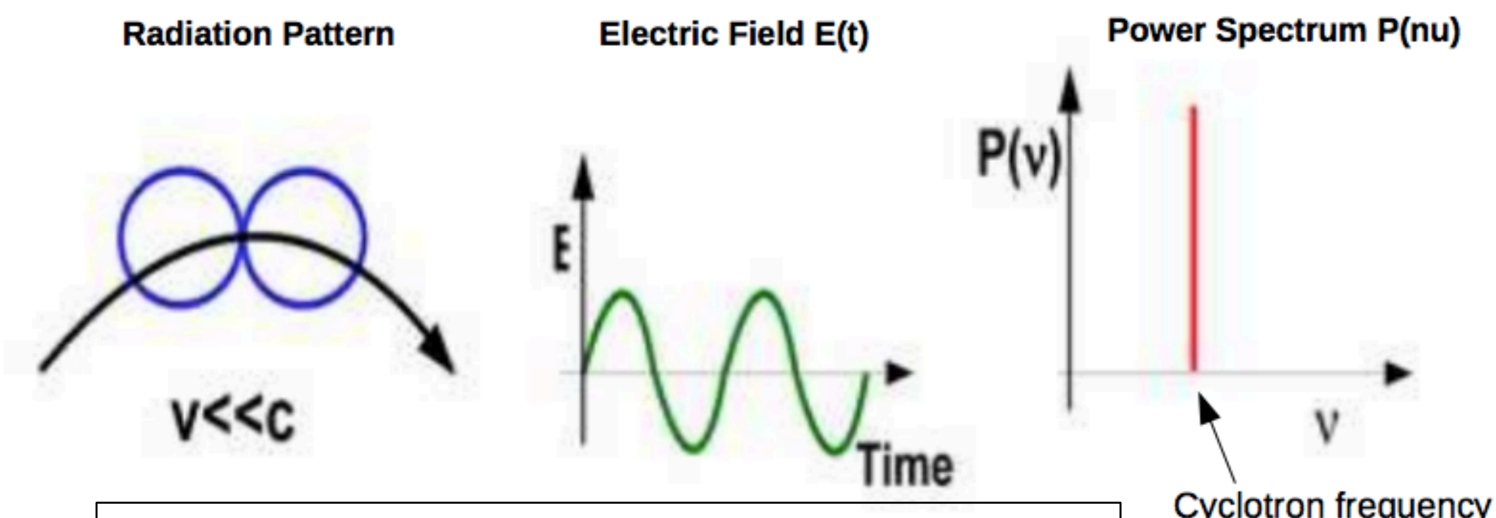
Frequency is independent of path radius and particle velocity

Cyclotron radiation

The emission appears at a single frequency

$$\nu_L=\omega_L/2\pi=qB/2\pi m = 2.8 \text{ MHz per Gauss}$$

The emission pattern is moving along the circle with constant velocity, the electric field measured will vary sinusoidally and the power spectrum will show a single frequency (the Larmor or cyclotron frequency).



Power spectra will peak at a single frequency

Cyclotron radiation

Kinetic energy

$(½)\ m\ v^2 = q\ V$

$v = \surd 2q\ V/m$

protons need to be much more energetic than electrons to become relativistic

$$r_L = \frac{mv}{qB} \rightarrow r_L = \sqrt{\frac{2mV}{qB^2}}$$

Calculate Larmor radius for 1 Mev proton in 1 Tesla (~10^4 Gauss) field

Calculate Larmor radius for 1 Mev electron in 1 Tesla (~10^4 Gauss) field

Table 7.1 The properties of protons, carbon and iron nuclei having Lorentz factors $\gamma = 2$ and 100.

	Proton		Carbon nucleus		Iron nucleus	
Lorentz factor, γ	2	100	2	100	2	100
Velocity, v	$(\sqrt{3}/2)\,c$	$0.99995\,c$	$(\sqrt{3}/2)\,c$	$0.99995\,c$	$(\sqrt{3}/2)\,c$	$0.99995\,c$
Mass number, A	1	1	12	12	56	56
Atomic number, z	1	1	6	6	26	26
Rest mass energy, mc^2	1 GeV	1 GeV	12 GeV	12 GeV	56 GeV	56 GeV
Total energy, γmc^2	2 GeV	100 GeV	24 GeV	1200 GeV	112 GeV	5600 GeV
Kinetic energy, $(\gamma - 1)mc^2$	1 GeV	99 GeV	12 GeV	1188 GeV	56 GeV	5544 GeV
Kinetic energy per nucleon	1 GeV	99 GeV	1 GeV	99 GeV	1 GeV	99 GeV
Momentum, $pc = (\gamma m\|\boldsymbol{v}\|)c^{\dagger}$	$\sqrt{3}$ GeV	99.995 GeV	20.8 GeV	1199.9 GeV	96.99 GeV	5599.7 GeV
Rigidity, pc/ze	$\sqrt{3}$ GV	99.995 GV	$2\sqrt{3}$ GV	199.99 GV	3.73 GV	215.4 GV

† To obtain the dimensions of GeV, the momentum has been multiplied by c, the velocity of light.

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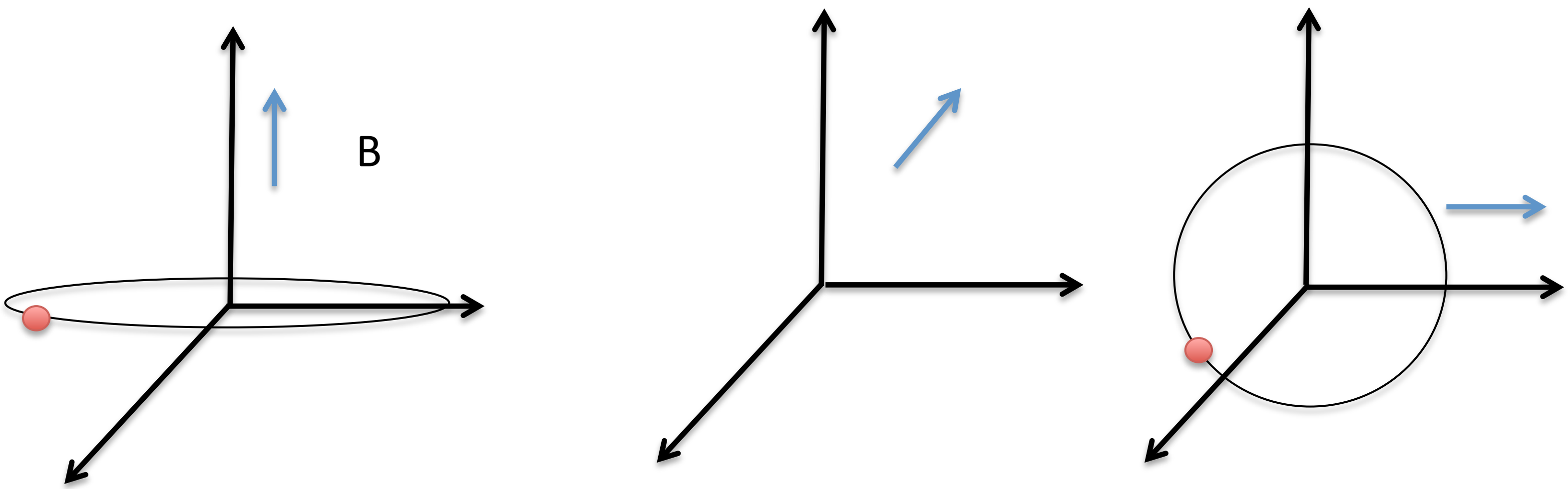
Cyclotron radiation

Polarization

B is perpendicular to LOS

B is at an angle to LOS

B is parallel to LOS



Linear Polarization

Elliptical Polarization

Circular Polarization

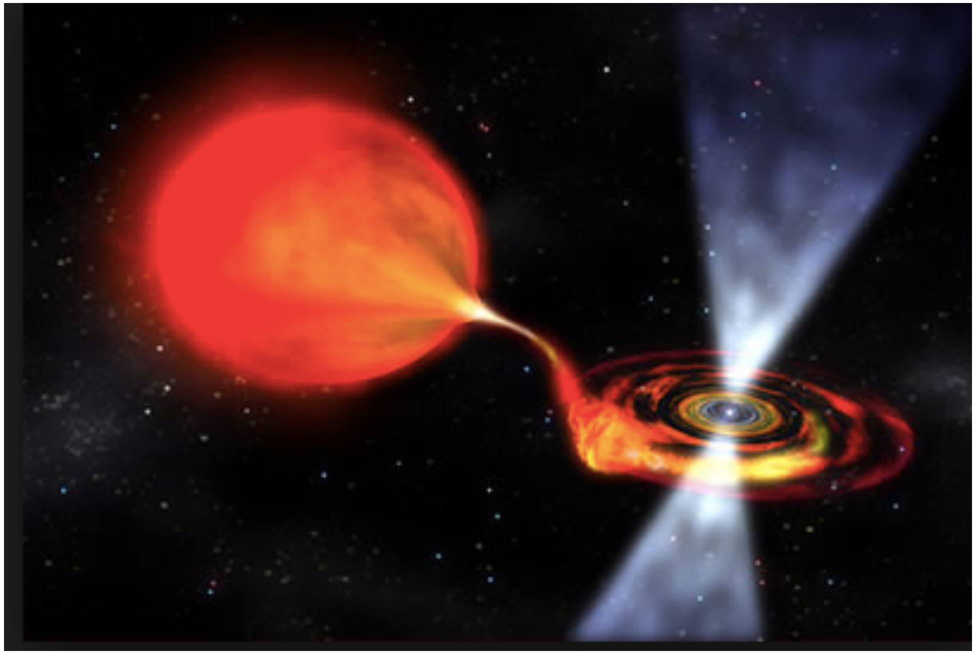
Polarization measurement to infer B strength and its orientation

Cyclotron radiation

Astrophysical application Cyclotron lines

Discovered ~ 40 years back

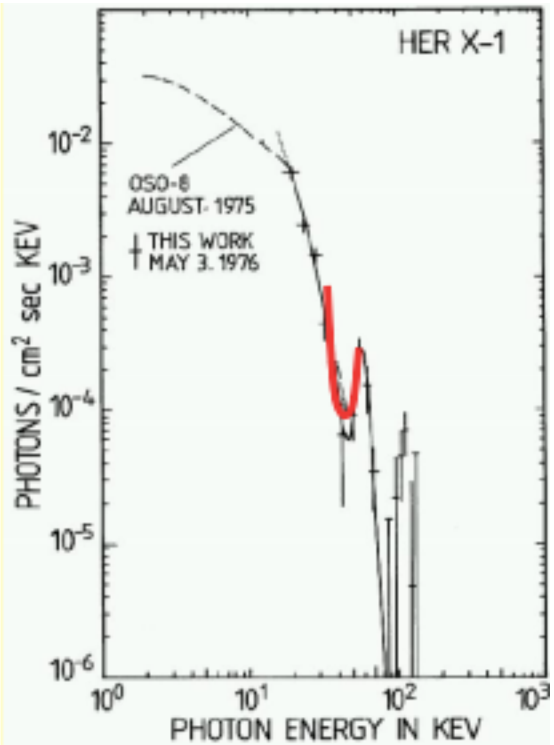
Cyclotron lines from the accreting X-ray pulsars

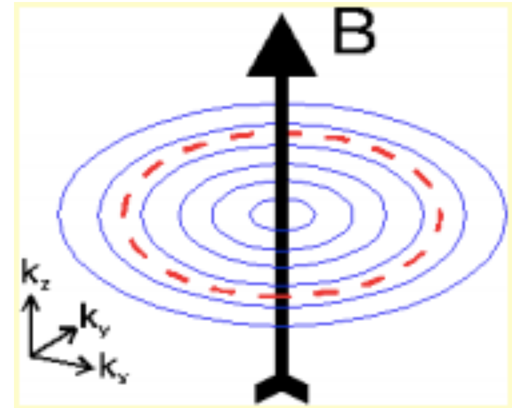


J. Trumper identified a cyclotron emission line in accreting pulsar Hercules X-1 @ 1977 : The X-ray spectrum shows an emission line at around 40 keV.

Trumper proposed : hot electrons around neutron star magnetic poles are rotating around a strong B field of ~ 5×10^{12} Gauss, giving rise to an emission line at ~ 40 keV.

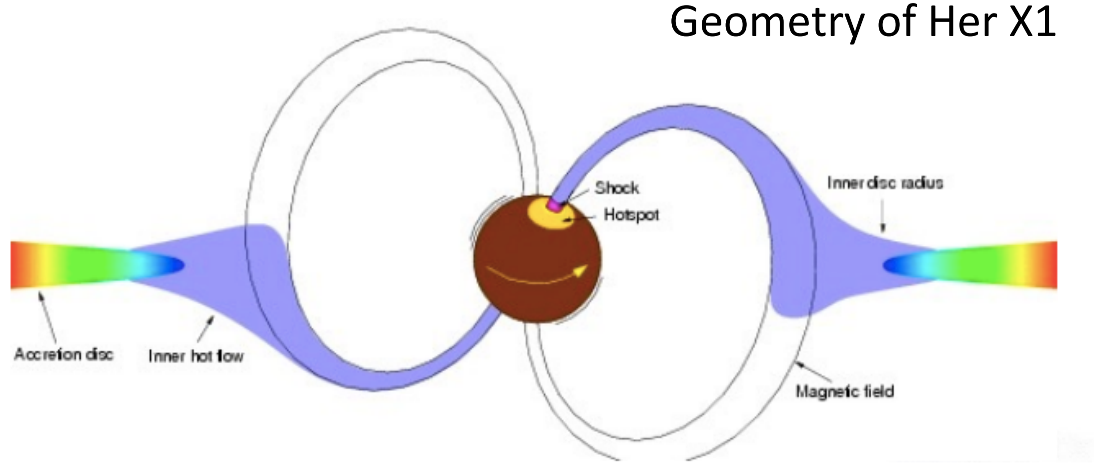
Cyclotron lines
Astrophysical application





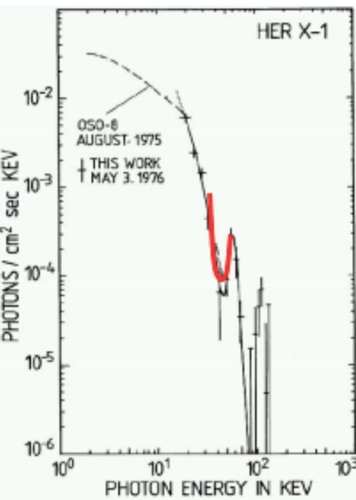
Directly probe the magnetic fields of the neutron stars
Probe geometry
Seen in more than 30 sources
Simulations + Observations

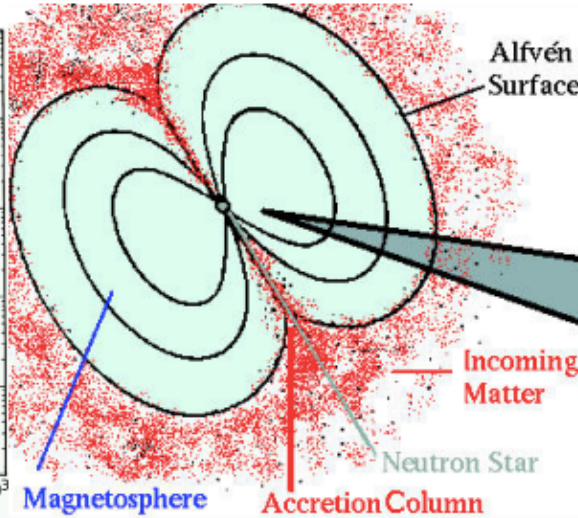
Geometry of Her X1

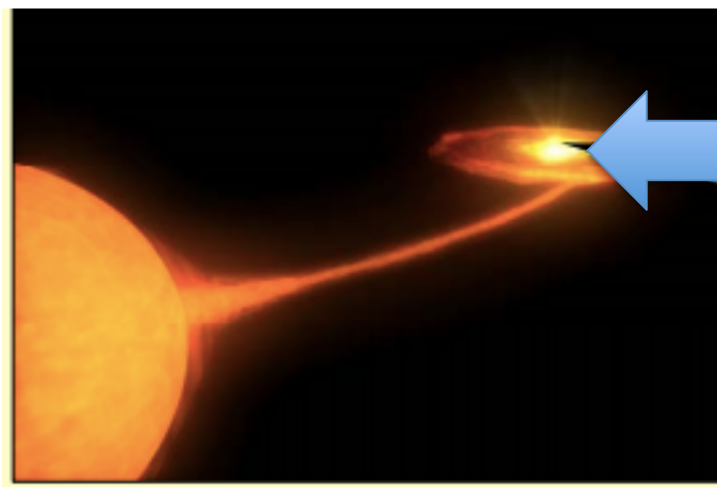


http://adsabs.harvard.edu/full/2003ASPC..308..271V

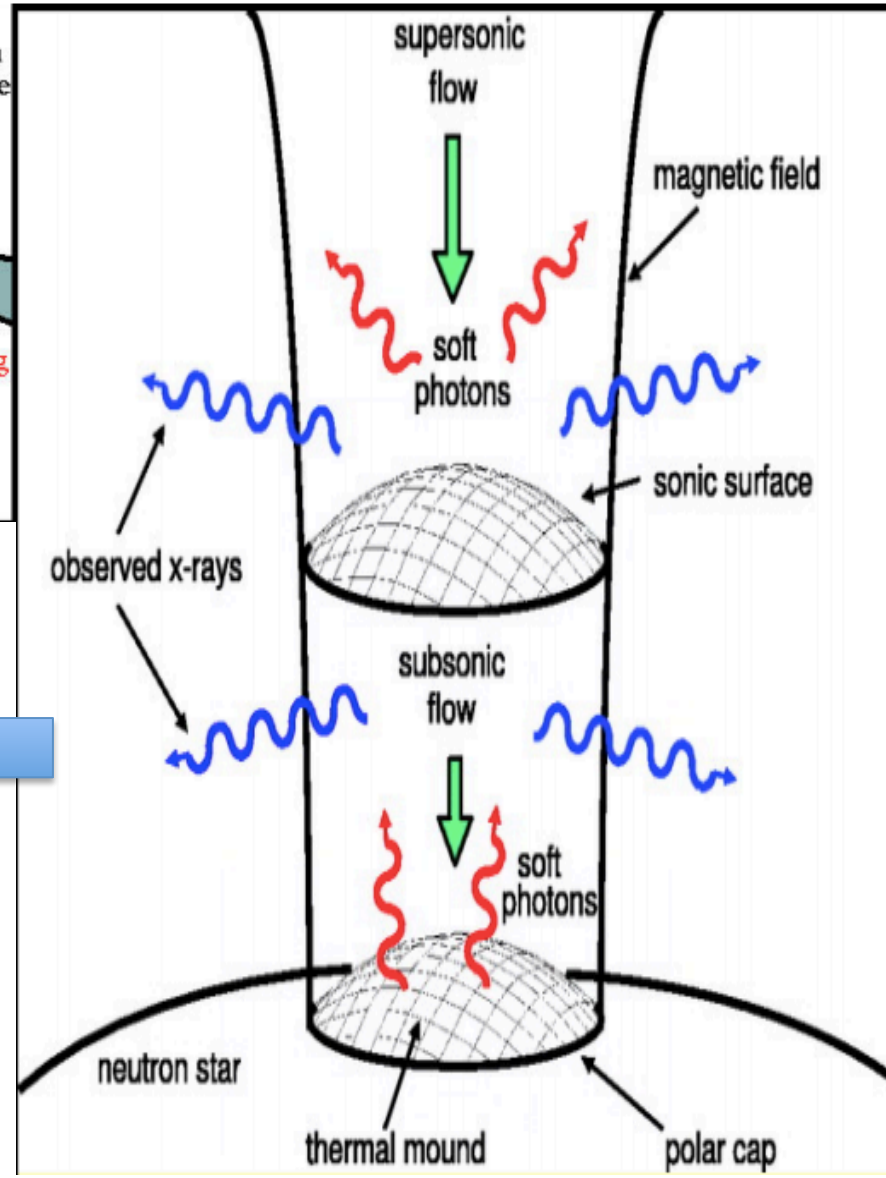
Observation







Modeling



Refer to :https://www.cosmos.esa.int/documents/13611/404108/200808_Schoenherr.pdf/ecff8c8e-f1e7-4f30-b3c8-66d682e20a13

Cyclotron radiation

Astrophysical application

Discovered ~ 40 years back

Cyclotron line analysis provides an elegant way to probe the magnetic field and the physics of accretion of neutron stars

Estimate the magnetic field if you get a cyclotron absorption feature at 34 KeV ?

Cyclotron radiation
Astrophysical application
Discovered ~ 40 years back

A substantial fraction of the known neutron stars reside in X-ray binaries, providing an ideal site to study these objects.

Neutron star binary systems/ accretion powered pulsars (ACPs), accrete matter from the companion and emit pulsed radiation at X-ray wavelengths.

Accretion powered X-ray pulsars are some of the most powerful sources of X-ray radiation in our Galaxy.
Luminosity within $10^{33} - 10^{35}$ erg s^{-1} during quiescence
Luminosity rise up to 10^{38} erg s^{-1} during active state
Strong magnetic fields up to 10^{11} - 10^{13} G

Cyclotron lines provided the first direct measurement of the magnetic field strength of a neutron star

Ref: Maitra et al 2017

Cyclotron radiation

Astrophysical application

Discovered ~ 40 years back

Cyclotron lines are usually detected as absorption lines in the continuum spectrum, and are modeled with Gaussian or pseudo-Lorentzian profiles

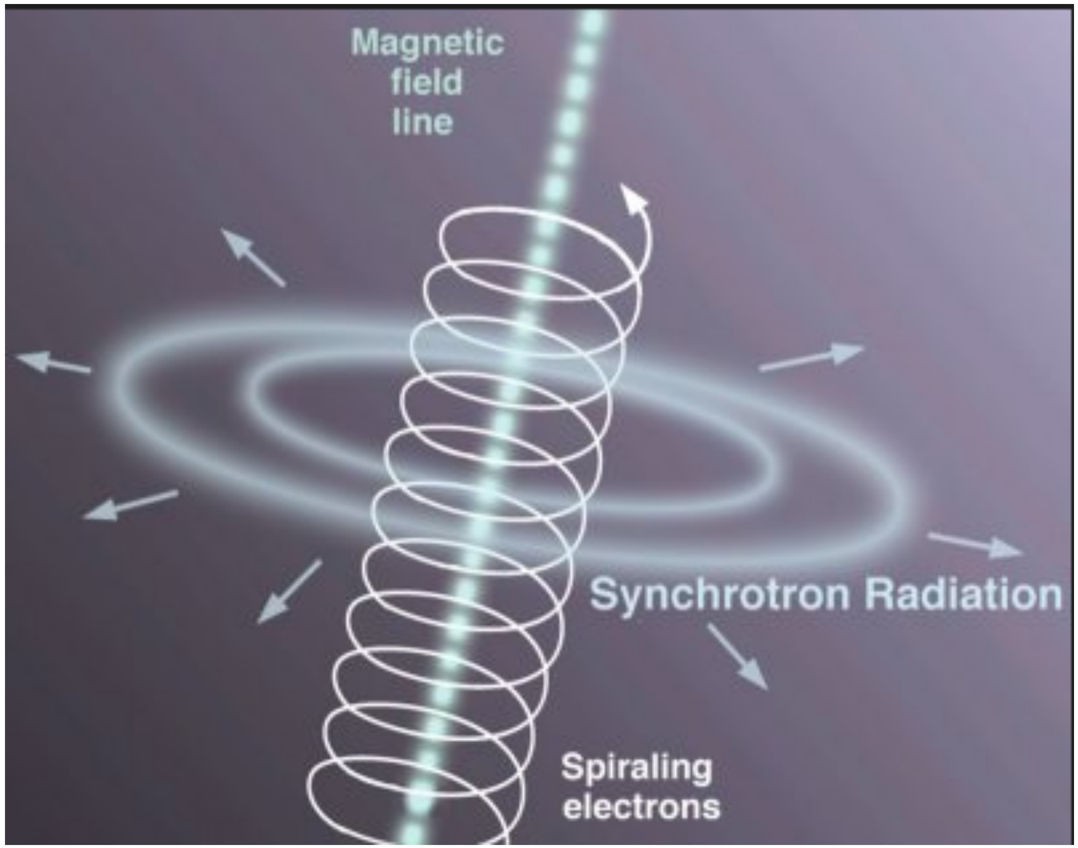
More than 30 such sources known

Changes of line parameters with luminosity provides probe of geometry

Ongoing missions NuSTAR and ASTROSAT are probing Cyclotron lines.

Modeling the timing and spectral results jointly with the latest physical models can provide a comprehensive picture on the physics of these accreting binary pulsar systems

Synchrotron Radiation is radiation emerging from a charge (moving relativistically) that is accelerated by a magnetic field.



Emission by ultra-relativistic electrons spiraling around magnetic field lines

Relativistic effects: from Cyclotron to Synchrotron Radiation

Assumption v<<c (non relativistic particles) **for Cyclotron**

Need to understand what happens to the radiation of a charge accelerated in a B field when the speeds approach c **for Synchrotron**

Review Relativistic effects discussed in Lecture 6

Lorentz transformations of time: $\Delta t = \Delta t' \gamma$

Lorentz transformations of Frequency: $\nu = \nu' / \gamma$

Relativistic effects: from Cyclotron to Synchrotron Radiation

Cyclotron → **Synchrotron**

Larmor Frequency → Frequency of Gyration

$$\nu_L=\omega_L/2\pi=qB/2\pi m$$

$$\nu_B=\omega_B/2\pi=qB/2\pi m\gamma$$

Larmor radius → Radius of Gyration

$$r_L=\frac{mv}{qB}\rightarrow r_L=\sqrt{\frac{2mV}{qB^2}}$$

$$r_L=\sqrt{\frac{(\gamma+1)mV}{qB^2}}=\frac{\gamma m v}{qB}$$

Period of rotation → Period of rotation

$$T=2\pi/\omega_L=2\pi m/qB$$

$$T=2\pi/\omega_B=2\pi m\gamma/qB$$

The period depend on particle velocity (Lorentz factor gamma) and as the velocity approaches c, the period increases.

Synchrotron Radiation

"Synchrotron" in synchrotron machines: the strength of the B field is not kept constant, but it is increased with time so that as gamma increases the frequency and the radius of gyration are constant.

Very famous Synchrotron machine : LHC (Large Hadron Collider)

↓

Synchrotron machine used to generate relativistic protons up to 7 TeV in energy (per beam).

Watch this video on LHC : https://www.youtube.com/watch?v=qQNpucos9wc

Synchrotron Radiation
In Astrophysics

Magnetic fields and relativistic particles are prerequisite for synchrotron radiation in astrophysics.

So synchrotron emission is seen in a wide variety of environments.

Typical magnetic field strengths

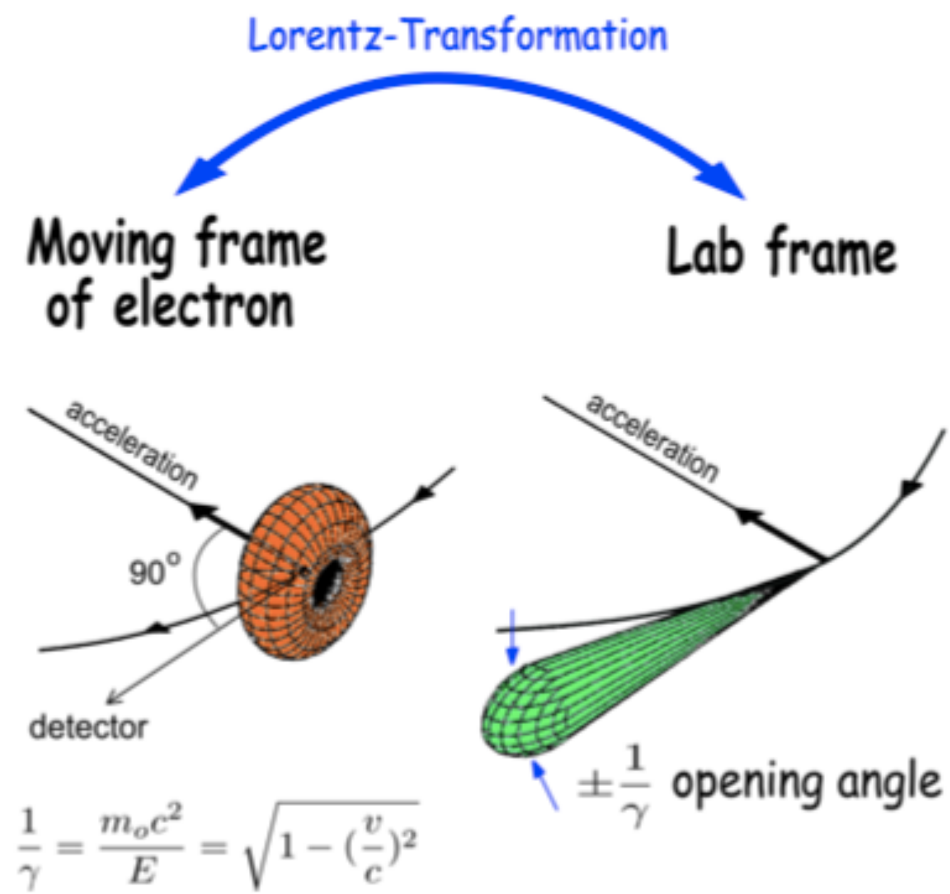
Location	Magnetic filed (Gauss)
Interstellar medium	10^{-6}
Stellar atmosphere	1
Black hole	10^{4}
White dwarf	10^{2}
Neutron star	10^{12}
Earth	0.3

Synchrotron Radiation
Emission pattern

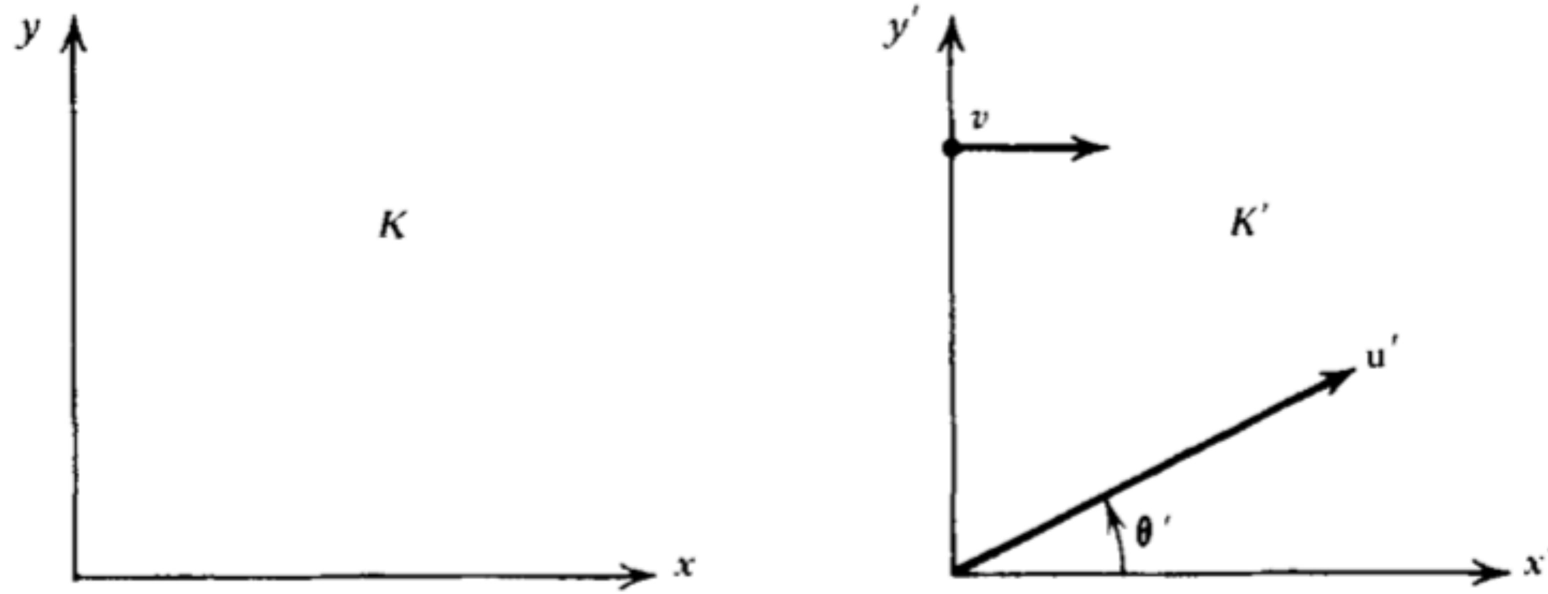
A relativistic electron moving around a B field.

Cyclotron to Synchrotron:

- start with the radiation pattern in the electron rest frame (where we know the radiation pattern)
- then we do a Lorentz transformation from the rest frame to the lab frame.



Transformations of velocities (Slide 11, Lecture 6)



$$u_{\parallel} = \frac{u'_{\parallel} + v}{\left(1 + v u'_{\parallel}/c^2\right)}, \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + v u'_{\parallel}/c^2\right)}$$

Components of **u** parallel and perpendicular to **v**

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + v)},$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)}$$

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}$$

Transformations of velocities
Beaming effect
(Lecture 7, slide 12)

For θ' = π/2, considering a photon emitted at right angles to v in K' ⟶

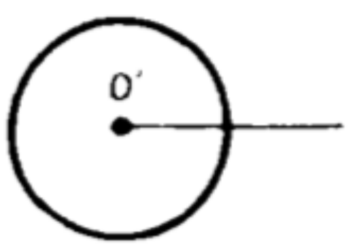
$$\tan\theta = \frac{c}{\gamma v}$$

$$\sin\theta = \frac{1}{\gamma}.$$

For highly relativistic speeds γ >>1 ⟶

$$\theta \sim \frac{1}{\gamma}$$

Consider photons are emitted isotropically in K'
In frame K the photons are concentrated in forward direction in a cone of 1/γ.
This is called **beaming effect**.



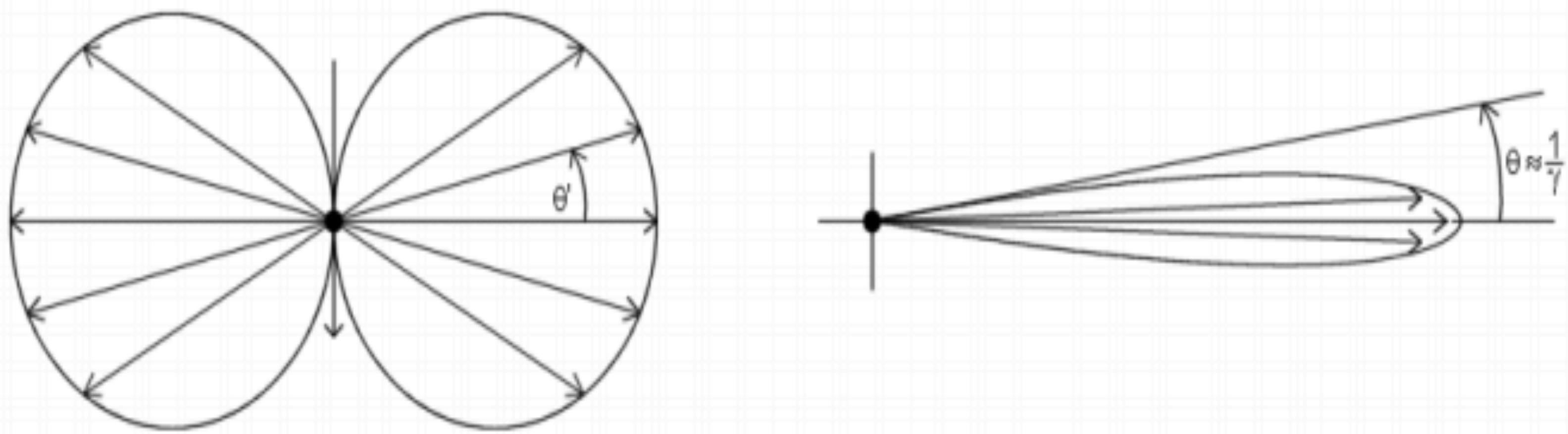
Isotropic emission: Rest frame K'

Beamed emission :K

Lorentz transformation

electron's own rest frame ⟷ laboratory frame of reference

$V_e << C$ $\qquad$ $V_e \approx C$



$$\tan\theta = \frac{\sin\theta'}{\gamma(\beta + \cos\theta')}$$

$$\theta \approx \frac{1}{\gamma}$$

half-opening angle in radians

In the rest frame, there is zero power emitted at angle $\theta' = \pi/2$, and so in the lab frame we have $\tan\theta = 1/\gamma$, which, for large γ, gives $\theta \approx 1/\gamma$. Thus all the forward power is radiated in a beam of angle $2/\gamma$.

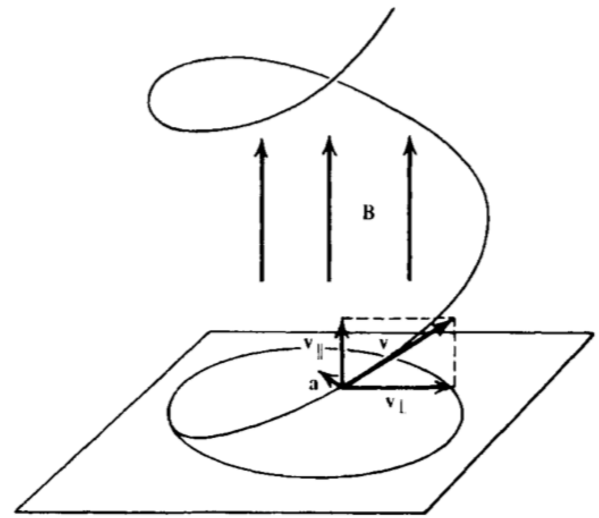
Synchrotron Radiation

Synchrotron radiation: Motion of ultra-relativistic particles around the magnetic field lines

Consider a particle of mass m and charge q

Equations of Motion of a particle with relativistic velocity:

Change of relativistic momentum dp/dt



$$\frac{d}{dt}(\gamma m\mathbf{v}) = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

γ is constant → |v| constant

$$m\gamma\frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Force on the particle is perpendicular to the motion.

Synchrotron Radiation

Helical Motion:

$$m\gamma \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v} \times \mathbf{B} \quad \longrightarrow \quad \frac{d\mathbf{v}_{\parallel}}{dt} = 0, \qquad \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{\gamma mc}\mathbf{v}_{\perp} \times \mathbf{B}$$

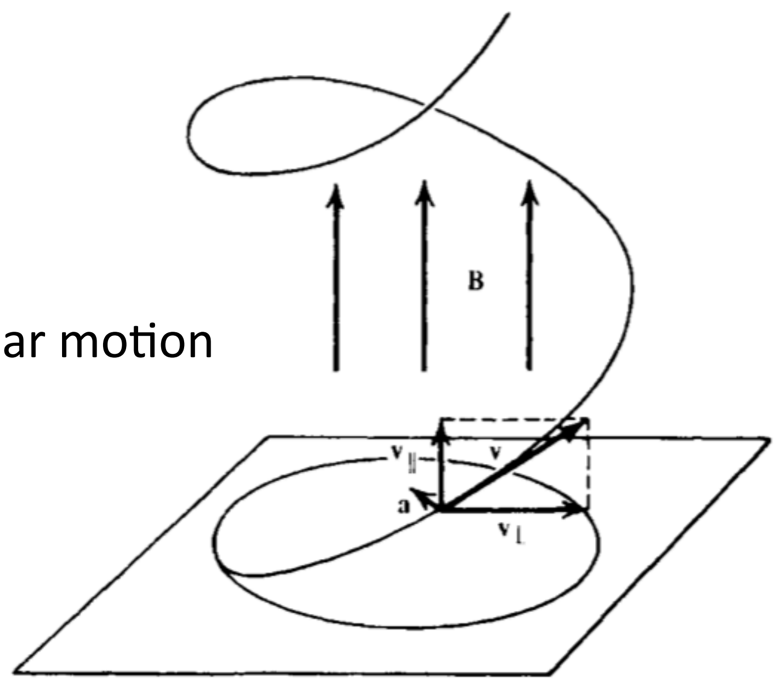
Separating the velocity components along the field and in a plane perpendicular to the field

Parallel component of v is constant
But |v|=constant
So perpendicular component of v is constant

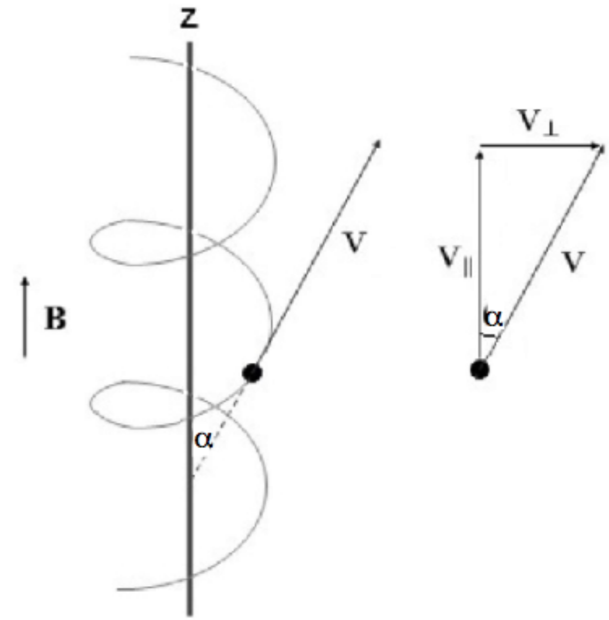
Motion of the particle is combination of circular motion
And uniform motion along the field

↓

Helical motion of the particle



Synchrotron Radiation



$$m\gamma \frac{d\mathbf{v}}{dt} = \frac{q}{c}\mathbf{v} \times \mathbf{B}$$ = Force = mv^2/r

r is the radius of the orbit ⟶ radius of gyration

α is angle between field and velocity ⟶ pitch angle

π/2 for motion perpendicular to fields

$$\omega_B = \frac{qB}{\gamma mc}$$

⬇

Calculate for different magnetic filed values (e.g. typical ISM, cosmic ray, neutron star etc)

Synchrotron Radiation

$$\omega_B = \frac{qB}{\gamma mc}$$

For ISM considering B ~ 10^{-6} G and γ=1 ⇨ ω_B ~30 Hz

Knowing the ω_B <1 Hz for cosmic ray electrons → estimate the field strength

Synchrotron Radiation
(Total power radiated)

Total emitted radiation (From Lecture 7)

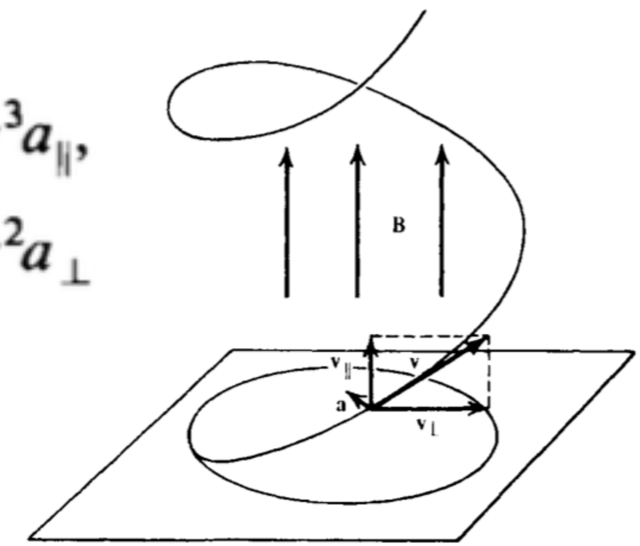
$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a'^2_{\parallel} + a'^2_{\perp}\right)$$

$$= \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

$$a'_{\parallel} = \gamma^3 a_{\parallel},$$

$$a'_{\perp} = \gamma^2 a_{\perp}$$

$a_{\perp} \rightarrow \omega_B v_{\perp}$; $a_{\parallel} \rightarrow$ zero



$$P = \frac{2q^2}{3c^3}\gamma^4 \frac{q^2 B^2}{\gamma^2 m^2 c^2} v_{\perp}^2 \longrightarrow P = \frac{2}{3} r_0^2 c \beta_{\perp}^2 \gamma^2 B^2$$

$$r_0 \equiv \frac{e^2}{mc^2}$$

Total emitted radiation from charged particles with velocity v

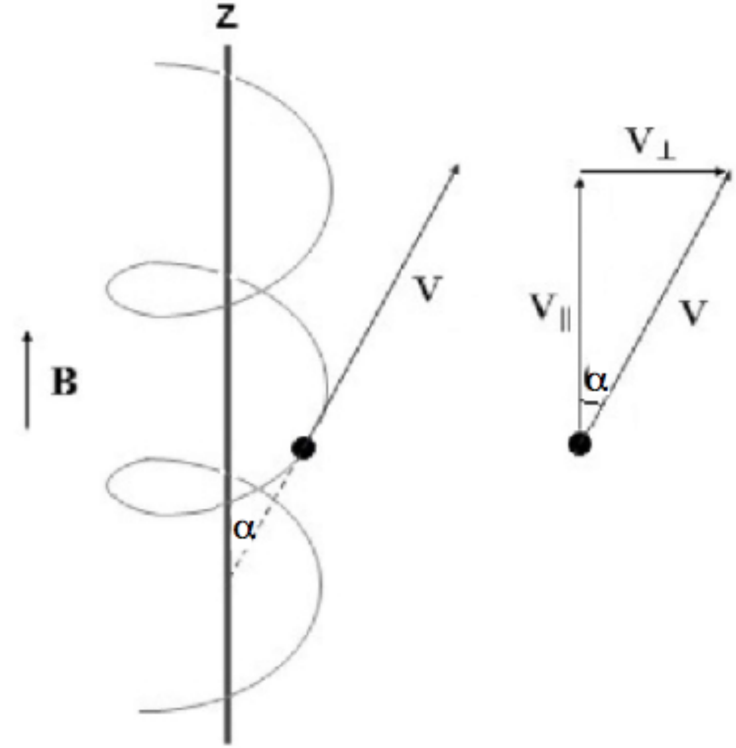
Synchrotron Radiation
(Total power radiated)

Total emitted radiation

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

We have many particles each having a pitch angle. So the perpendicular velocity needs to be averaged over all pitch angles (α).

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \, d\Omega = \frac{2\beta^2}{3}$$



Synchrotron Radiation
(Total power radiated)

Total emitted radiation

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

We have many particles each having a pitch angle. So the perpendicular velocity needs to be averaged over all pitch angles (α).

$$\langle \beta_\perp^2 \rangle = \frac{\beta^2}{4\pi} \int \sin^2\alpha \, d\Omega = \frac{2\beta^2}{3}$$

Total emitted radiation

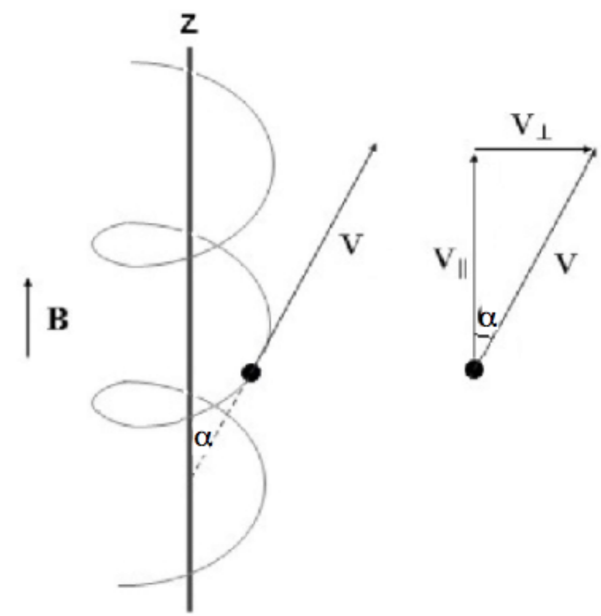
$$P = \left(\tfrac{2}{3}\right)^2 r_0^2 c \beta^2 \gamma^2 B^2,$$

$$\sigma_T = 8\pi r_0^2/3$$

$$U_B = B^2/8\pi$$

Total emitted radiation for electrons

$$P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B$$



Synchrotron Radiation

(Total power radiated)

$$P = \frac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2$$

→ Valid for electron only

$$P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B$$

→ Valid for electron only

The formula is valid only for electrons emitting synchrotron radiation.

The reason why we write this formula only for electrons is because in basically all astrophysical cases you have electron synchrotron.
This is because electrons become relativistic much more quickly than protons as they are easier to accelerate.

Synchrotron Radiation
(Total power radiated)

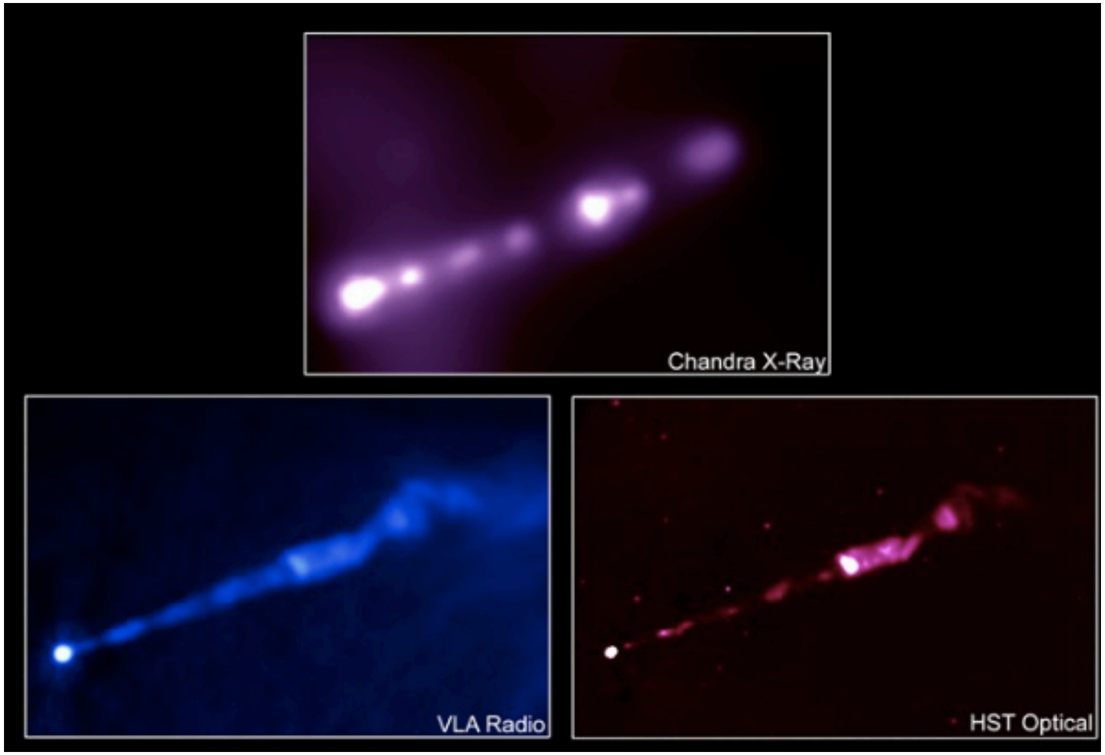
Suppose the protons of the LHC are accelerated up to an energy of 7 TeV and then they are left to cool down due to synchrotron emission.
On which timescale do they cool down?

Time scale ~ (Proton energy)/(Synchrotron power) ~ few days

Time scale ~ (Electron energy)/ (Synchrotron power) ~ nano seconds

Electrons cools down by a factor of ~10^{13} times faster than protons

Synchrotron in Astrophysics

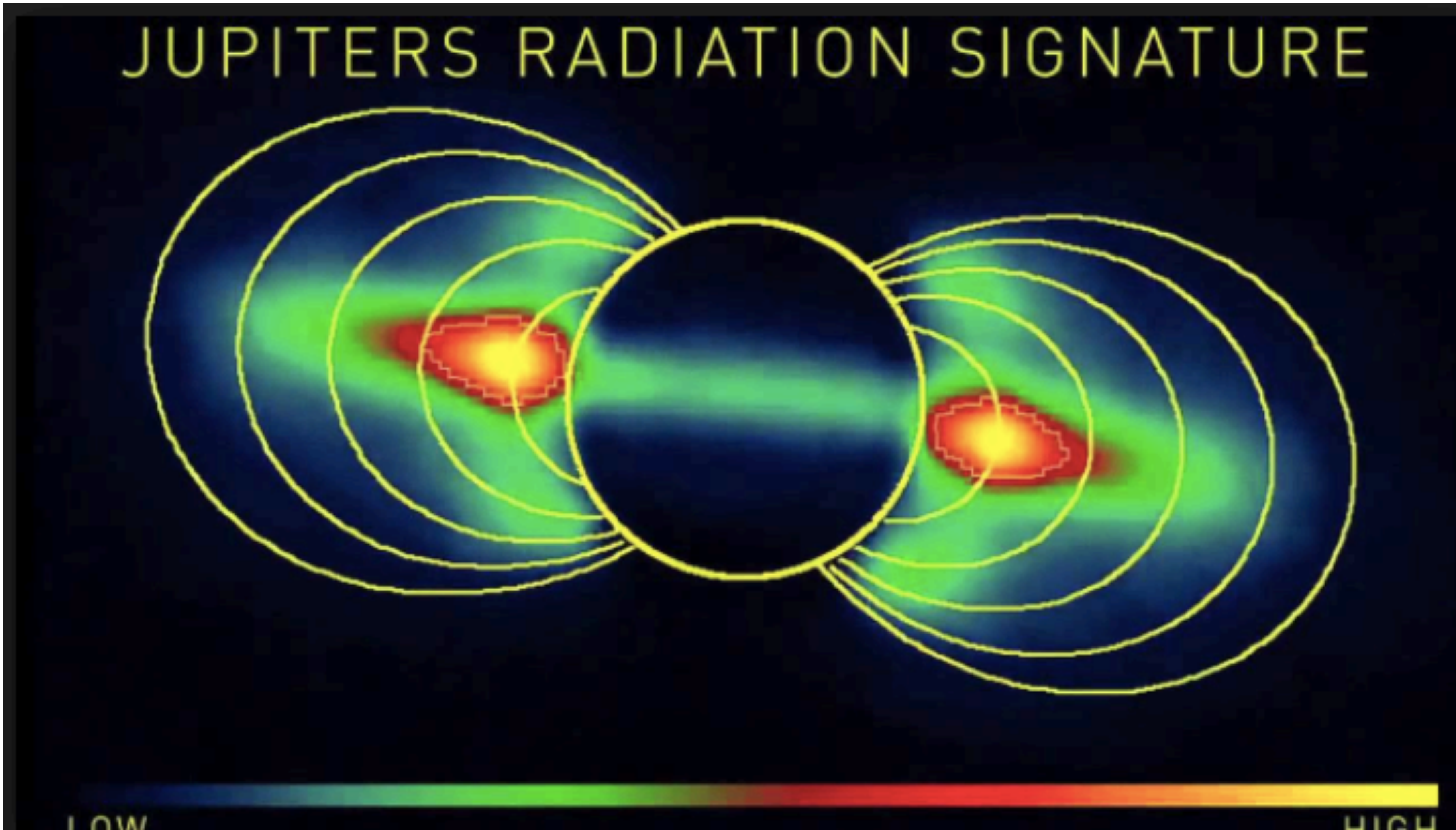


Astrophysical Jets

Courtesy :Alessandro Patruno

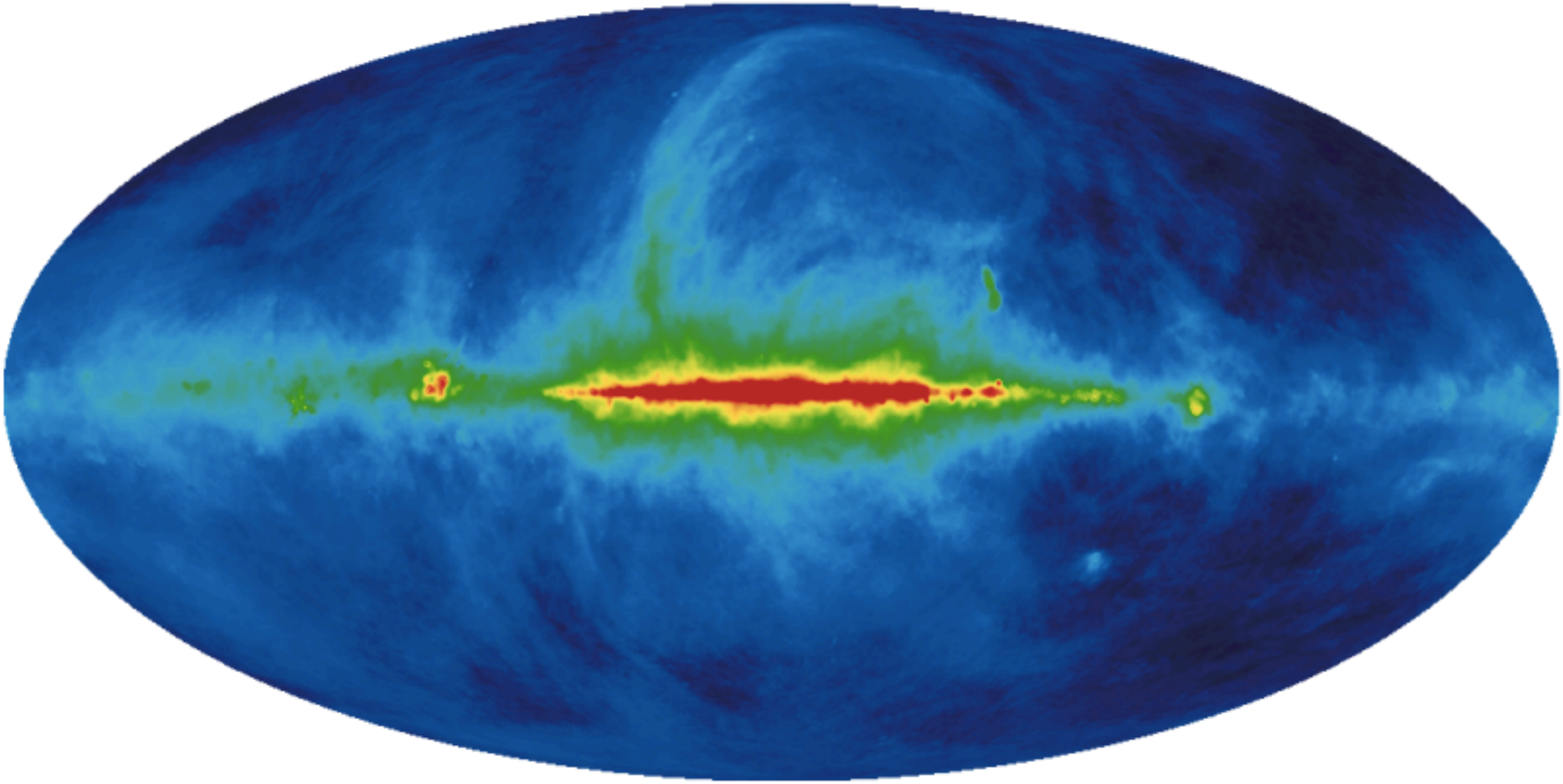
"Astrophysical jets are most likely generated by relativistic particles being launched close to a black hole (or even a neutron star when in a binary). Such particles are thought to be electron/positron pairs which then spiral along B field lines and generate synchrotron radiation. However, we also know that cosmic rays most likely come from Active Galactic Nuclei, where strong B fields around supermassive black holes launch streams of ultra-relativistic particles which include protons. So it's still unclear whether jet emission is due to leptons or hadrons."

Synchrotron Radiation
Jupiter's Belt



Galactic Synchrotron

Haslam et al. map at 408 MHz for Galactic synchrotron emission



End of Lecture 9

Reference : Rybicki & Lightman Chapter 6

http://demonstrations.wolfram.com/SynchrotronRadiation/

Next Lecture : 9th September

Topic of next Lecture:
Synchrotron Radiation
(Chapter 6 of Rybicki & Lightman)