# Electrodynamics and Radiative Processes I

# Lecture 4 – Basic theory of radiation fields & Radiation from moving charges

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# Lecture -4

## Questions raised in the class

Why do not we see HI emission in earth, what fraction spin up what fraction down?

Einstein's coefficients are valid when thermodynamic equilibrium is not present?

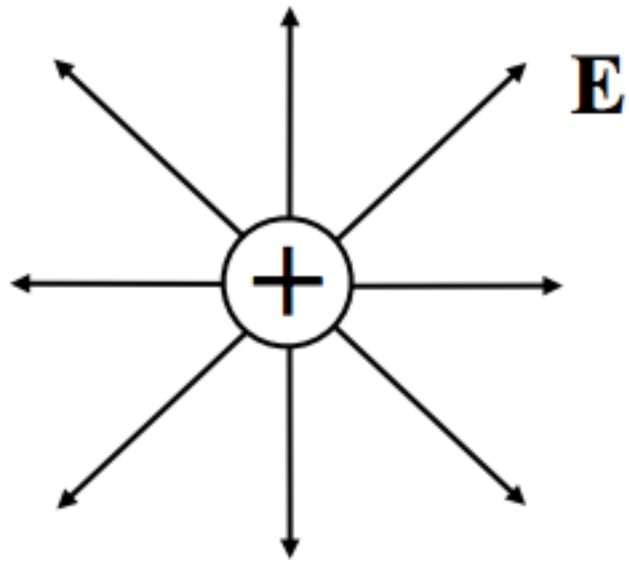
Are the relation between Einstein coefficients truly independent of temperature?

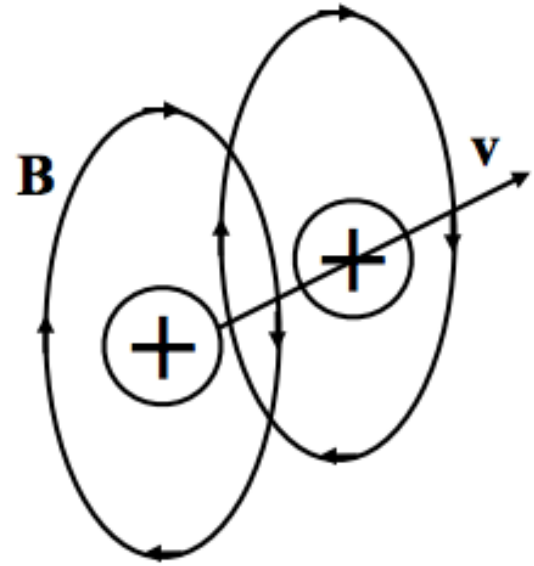
Radiation pressure derivation in Rybicki and lightman's textbook.
Consider only the solid angle of $2\pi$ instead of $4\pi$. Why?

Can the proton flip the spin instead of the electron (regarding 21cm line)?
Reserve for later

# Electric and magnetic field

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$




Static charge → Electric field
Force on a test charge

Moving charge → Magnetic field
→ Electric current
Magnetic force on a moving charge

Field of force @ Maxwell

Space around electrified object @ Faraday

Units → Newton/Coulomb
→ volts per m

Units → Newton/(Coulomb m/s)
→ Tesla
→ Gauss

# Electromagnetic flux

Lorentz force $\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$

Rate of work done $\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \mathbf{E}$,

Current density $\mathbf{j} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i$,

Rate of work done $\frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}$.

Rate of change in mechanical energy of system per unit volume $\frac{dU_{\text{mech}}}{dt} = \mathbf{j} \cdot \mathbf{E}$.

# Maxwell's Equations
(in Gaussian units)

$$\nabla \cdot \mathbf{D} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

# Maxwell's Equations

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$$\mathbf{D} = \epsilon \mathbf{E},$$

$$\mathbf{B} = \mu \mathbf{H},$$

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

# Electromagnetic flux

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]$$

Using the following relation

$$\mathbf{E}\cdot(\nabla\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\nabla\cdot(\mathbf{E}\times\mathbf{H}),$$

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-c\nabla\cdot(\mathbf{E}\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]$$

# Electromagnetic flux

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Poynting's theorem

$$\mathbf{j}\cdot\mathbf{E}+\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2+\frac{B^2}{\mu}\right)=-\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right),$$

↓ Rate of change of Mechanical energy

↓ Rate of change of field energy $U_E+U_B$

↓ Divergence of field energy flux

# Electromagnetic flux

Electromagnetic field energy per unit volume

$$U_{\text{field}} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = U_E + U_B$$

Electromagnetic flux vector or Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.$$

# Plane electromagnetic waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

How ?

Wave equation with B ?

# Plane electromagnetic waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Solutions of wave equation

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\mathbf{B} = \hat{\mathbf{a}}_2 B_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

wave vector

frequency

unit vectors

complex constants

# Plane electromagnetic waves

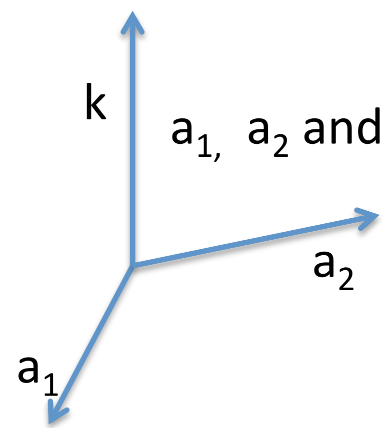
Substitution in Maxwell's equation

$$i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0 = 0 \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 = 0$$

$a_1$ and $a_2$ are perpendicular to k

$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 = \frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 = -\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0.$$

$a_1$ and $a_2$ are perpendicular




$a_1$, $a_2$ and k are perpendicular

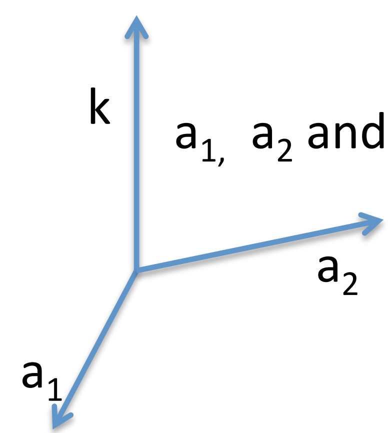
# Plane electromagnetic waves

Substitution in Maxwell's equation

$$i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0 = 0 \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 = 0$$

$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 = \frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 = -\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0.$$

$$E_0 = \frac{\omega}{kc} B_0, \qquad B_0 = \frac{\omega}{kc} E_0,$$

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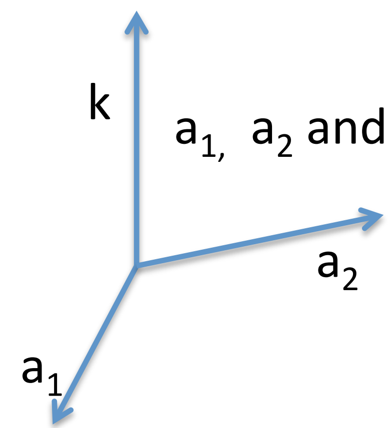
$$E_0 = \frac{\omega}{kc} B_0, \qquad B_0 = \frac{\omega}{kc} E_0,$$

$$E_0 = \left(\frac{\omega}{kc}\right)^2 E_0 \Rightarrow \omega = ck.$$

$$E_0 = B_0. \qquad v_{\mathrm{ph}} = c. \Rightarrow$$

EM waves travels at speed of light




$a_1$, $a_2$ and k are perpendicular

# Plane electromagnetic waves
## Energy flux and energy density

Time averaged pointing vector $\langle S \rangle = \frac{c}{8\pi}\mathrm{Re}(E_0 B_0^*)$.

Since $E_0 = B_0$.

$$\langle S \rangle = \frac{c}{8\pi}|E_0|^2 = \frac{c}{8\pi}|B_0|^2$$

Similarly time averaged energy density

$$\langle U \rangle = \frac{1}{16\pi}\mathrm{Re}(E_0 E_0^* + B_0 B_0^*)$$

$$\langle U \rangle = \frac{1}{8\pi}|E_0|^2 = \frac{1}{8\pi}|B_0|^2$$

# Plane electromagnetic waves
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# Radiation spectrum

$$\Delta\omega\Delta t > 1$$

$\downarrow$

with a radiation field of length Δt we can define spectrum with in Δω

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt.$$

Energy per unit area per unit time

$$\frac{dW}{dt\,dA} = \frac{c}{4\pi}E^2(t)$$

Total Energy per unit area

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt$$

# Radiation spectrum

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega.$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 4\pi \int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega.$$

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)\,dt \longrightarrow \frac{dW}{dA} = c\int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega,$$

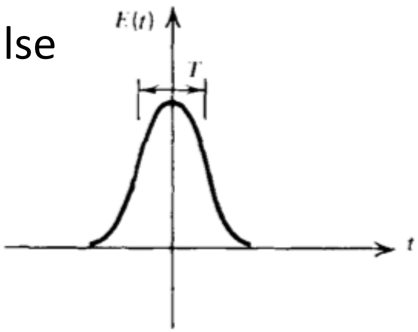
Energy per unit area per unit frequency $\longrightarrow$ $\dfrac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2$

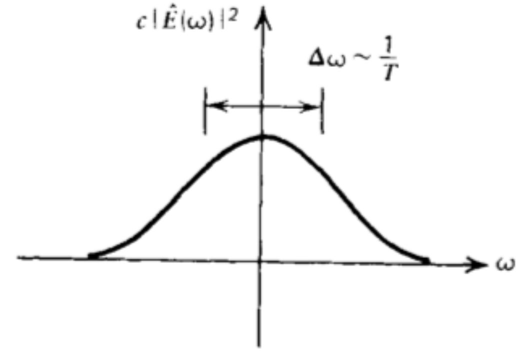
# Radiation spectrum

Electric field

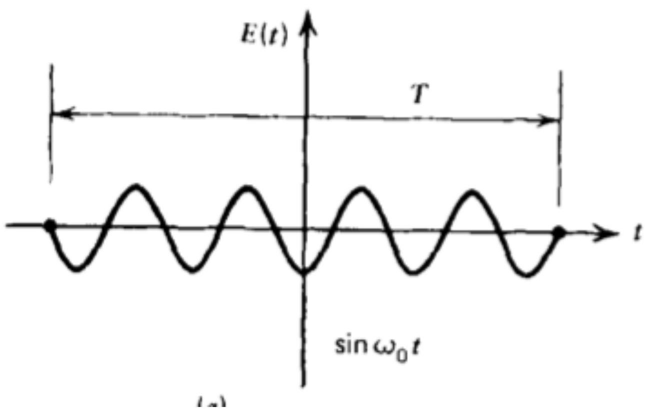
Power spectrum

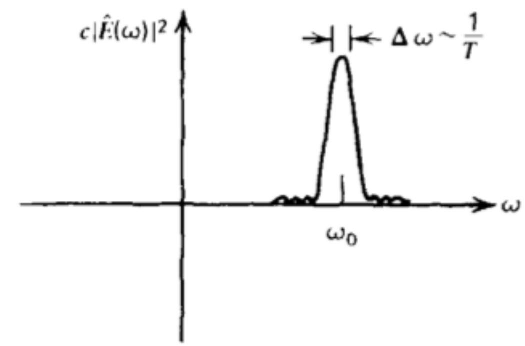
a) pulse




b) sinusoidal pulse




Time extent of pulse T determines width of finest features : $\Delta\omega \sim 1/T$

Sinusoidal time dependence in pulse shape causes spectrum concentrated near $\omega \sim \omega_0$

# Electromagnetic Potentials

**E** and **B** are replaced by Φ(r,t) and **A(r,t)**

Why we need EM potentials?

1) One scalar plus one vector simpler than two vectors

2) Determining **A** and Φ are simpler

3) Relativistic EM theory will be simpler

# Electromagnetic Potentials

E and B are replaced by Φ(r,t) and **A(r,t)**

Why we need EM potentials?

1) One scalar plus one vector simpler than two vectors

2) Determining **A** and Φ are simpler

3) Relativistic EM theory will be simpler

Maxwell's equation $\nabla \cdot \mathbf{B} = 0$

Vector potential **A(r,t)** defined as $\mathbf{B} = \nabla \times \mathbf{A}.$

Thus

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \longrightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Scalar potential Φ(r,t) defined as $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$

# Electromagnetic Potentials

E and B are replaced by Φ(r,t) and **A(r,t)**

$$\mathbf{E}+\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}=-\nabla\phi$$

⇓

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Remember

$$\nabla\cdot\mathbf{D}=4\pi\rho$$

$$\nabla^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-4\pi\rho$$

# Electromagnetic Potentials

Thus from Maxwell's equations,

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

# Electromagnetic Potentials

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Thus from Maxwell's equations,

$$\nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$\Downarrow$

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

# Electromagnetic Potentials

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Thus from Maxwell's equations,

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$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j}$$

# Electromagnetic Potentials

Thus from Maxwell's equations,

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

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$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j}$$

# Electromagnetic Potentials

Scalar and vector potential are not uniquely determined by the conditions

For example, the addition of gradient ψ to **A** will not change **B**

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \psi, \qquad \mathbf{B} \rightarrow \mathbf{B}.$$

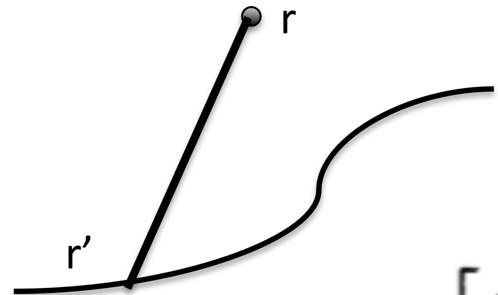
Electric field will not change if ϕ is changed in following manner

$$\phi \rightarrow \phi - \frac{1}{c} \frac{\partial \psi}{\partial t}, \qquad \mathbf{E} \rightarrow \mathbf{E}.$$

Such alterations of **A** and φ are called Gauge transformation

| | |
|---|---|
| Lorentz Gauge | $\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$ |

# Retarded Potentials




$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \quad \longrightarrow \quad \phi(\mathbf{r},t) = \int \frac{[\rho]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \quad \longrightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{[\mathbf{j}]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$[Q] \equiv Q\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)$$

Information from point r’ propagates at speed of light.

↓

The potential at r can only be affected by conditions at r’ at a retarded time t-|r-r’|/c

# Retarded Potentials

✓ Retarded time refers to conditions at the point r' that existed at a time earlier than t by time required for light to travel between r and r'

✓ Information from point r' propagates at speed of light, so potential at r can be affected by conditions of r' at this retarded time

✓ Solutions with advanced time are not permitted physically

For given charge and current density first find the retarded potentials and then determine **E** and **B**

# Retarded potential of single moving charges : Lienard-Wiechart potentials

Particle of charge q moving along a trajectory $r=r_0(t)$, velocity $u(t)=\dot{r}_0(t)$

Charge and current density

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{r}_0(t))$$

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

$$q = \int \rho(\mathbf{r},t)\,d^3\mathbf{r}$$

$\downarrow$

Total Charge

$$q\mathbf{u} = \int \mathbf{j}(\mathbf{r},t)\,d^3\mathbf{r}$$

$\downarrow$

Total Current

# Reterded potential of single moving charges : Lienard-Wiechart potentials

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Charge and current density

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$$\mathbf{j}(\mathbf{r},t)=q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

$$\boxed{\phi(\mathbf{r},t)=\int\frac{[\rho]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}}$$ → Scalar Potential

$$[Q]\equiv Q\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)$$

$$\phi(\mathbf{r},t)=\int d^3\mathbf{r}'\int dt'\,\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

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$$\phi(\mathbf{r},t)=\int d^3\mathbf{r}'\int dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)$$

↓ $$q=\int\rho(\mathbf{r},t)\,d^3\mathbf{r}$$

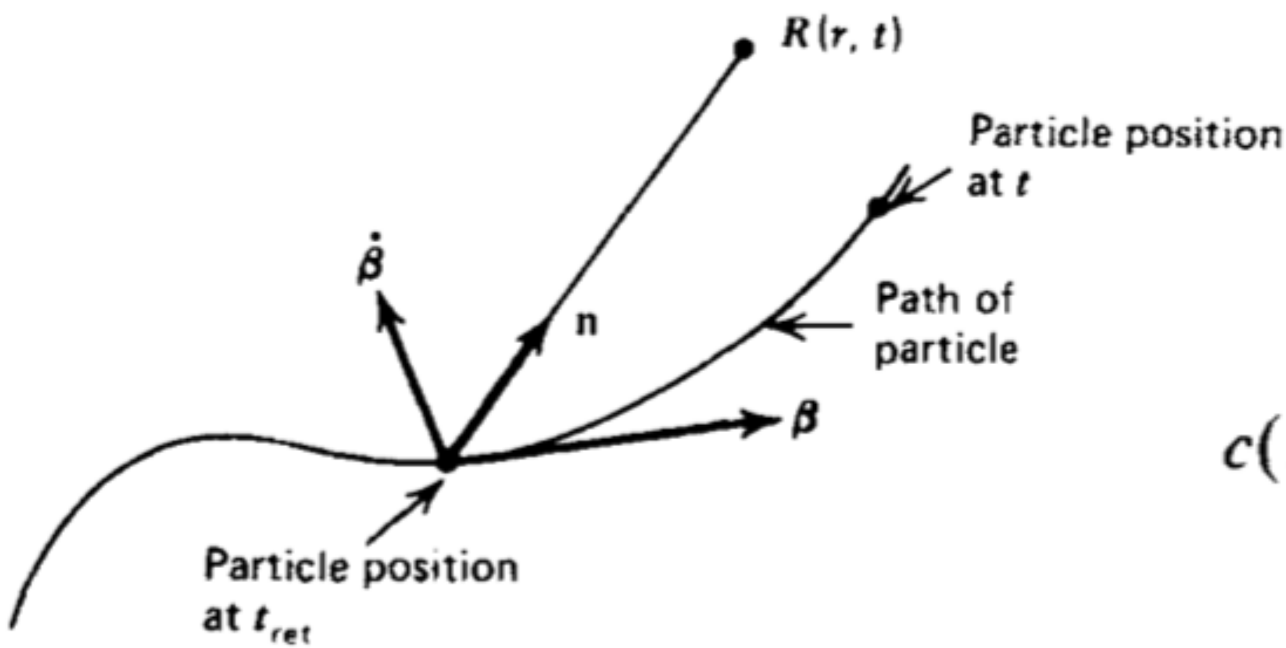
$$\phi(\mathbf{r},t)=q\int\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi(\mathbf{r},t)=q\int \delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}$$

$$\mathbf{R}(t')=\mathbf{r}-\mathbf{r}_0(t')$$

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\delta(t'-t+R(t')/c)dt'$$




δ function

A value of $t'=t_{ret}$ given by,

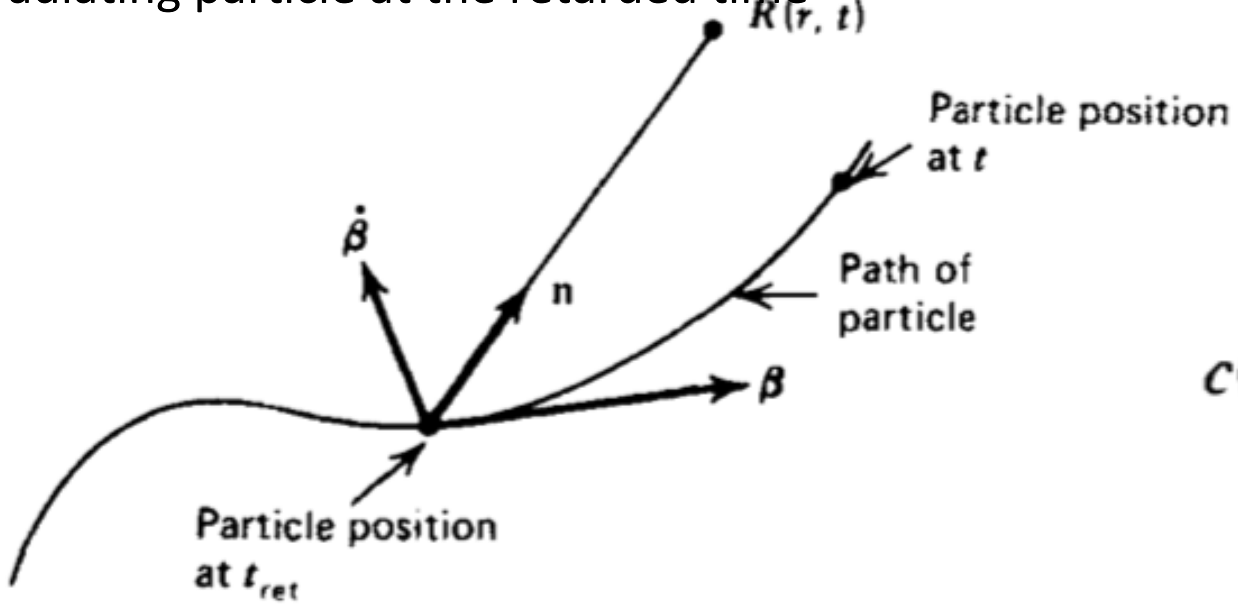
$$c(t-t_{ret})=R(t_{ret})$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\delta(t'-t+R(t')/c)dt'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{q}{c}\int \mathbf{u}(t')R^{-1}(t')\delta(t'-t+R(t')/c)dt'$$

Fig : Radiation field at R from position of the radiating particle at the retarded time




δ function

A value of $t'=t_{ret}$ given by,

$$c(t-t_{ret})=R(t_{ret}).$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

Change the variable from t' → t''

$$t'' = t' - t + [R(t')/c],$$

$$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'.$$

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\kappa(t') = 1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

Change the variable from t' → t''

$$t'' = t' - t + [R(t')/c],$$

$$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'.$$

$$R^2(t') = \mathbf{R}^2(t')$$

$$2R(t')\dot{R}(t') = -2\mathbf{R}(t')\cdot\mathbf{u}(t'),$$

$$\mathbf{n} = \frac{\mathbf{R}}{R}$$

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\kappa(t') = 1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

Change the variable from t' → t''

$$t'' = t' - t + [R(t')/c],$$

$$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'.$$

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\phi(\mathbf{r},t) = q\int R^{-1}(t')\left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]^{-1}\delta(t'')dt''$$

$$\phi(\mathbf{r},t) = \frac{q}{\kappa(t_{\text{ret}})R(t_{\text{ret}})} \qquad \kappa(t') = 1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')$$

Lienard-Wiechart Potential

$$\phi = \left[\frac{q}{\kappa R}\right] \qquad \mathbf{A} = \left[\frac{q\mathbf{u}}{c\kappa R}\right].$$

# Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi = \left[ \frac{q}{\kappa R} \right] \qquad \mathbf{A} = \left[ \frac{q\mathbf{u}}{c\kappa R} \right]$$

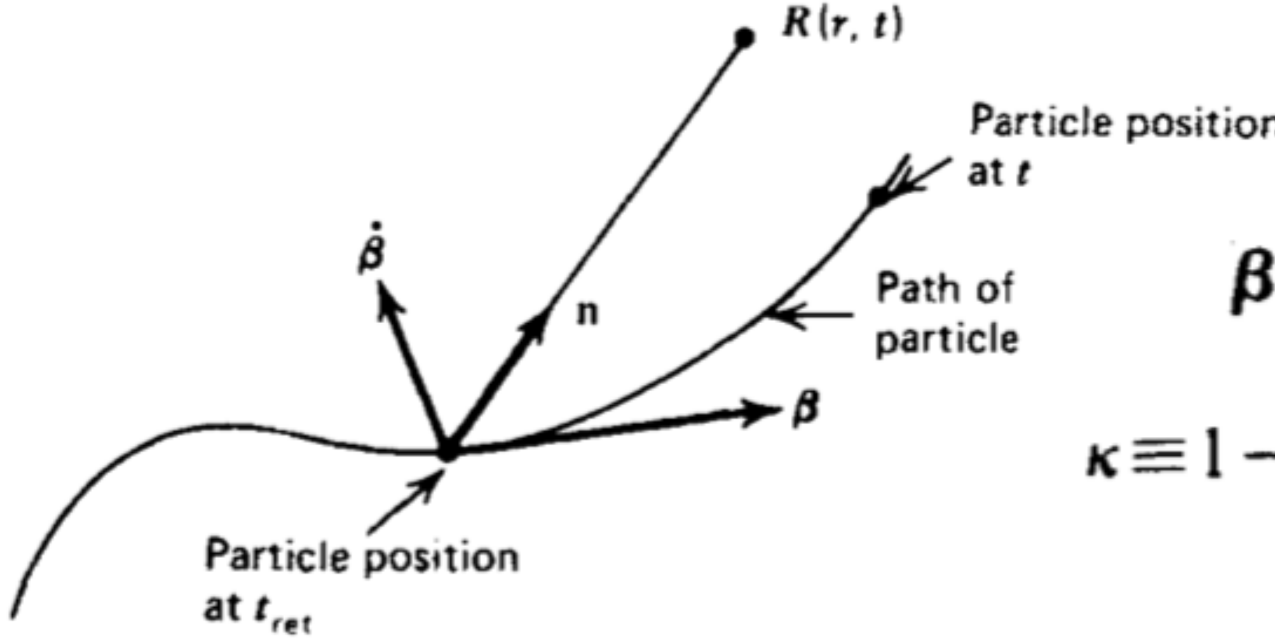
Differ from static electromagnetic theory in two ways

1) Extra factor κ : Important for velocities close to light.
   Tends to concentrate/beam potential into a narrow cone about particle velocity.
   Beaming effect (will be detailed in coming lectures)

2) Quantities are evaluated at retarded time.

Differentiate the potentials to get electric field(E) and magnetic field(B) (Jackson Section 14)

# Velocity and Radiation field




$$\beta \equiv \frac{\mathbf{u}}{c}$$

$$\kappa \equiv 1 - \mathbf{n} \cdot \boldsymbol{\beta}.$$

Fig : Radiation field at R from position of the radiating particle at the retarded time

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}(\mathbf{r},t) = [\mathbf{n} \times \mathbf{E}(\mathbf{r},t)]$$

Velocity field

Acceleration/Radiation field

# Radiation field

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

Velocity field

- $1/R^2$ dependence
- Only contributing term for particle with constant velocity
- Generalization of the Coulomb's law to moving particles, approaches to coulomb's law when u<<c
- Electric filed always point towards current position of the particle

Acceleration field/Radiation field

1/R dependence

Proportional to particle's acceleration

perpendicular to n

# Radiation field

$$\mathbf{E}(\mathbf{r},t)=q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3R^2}\right]+\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

Radiation field

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}_{\mathrm{rad}}(\mathbf{r},t)=[\mathbf{n}\times\mathbf{E}_{\mathrm{rad}}]$$

$E_{rad}$, $B_{rad}$, n : mutually perpendicular

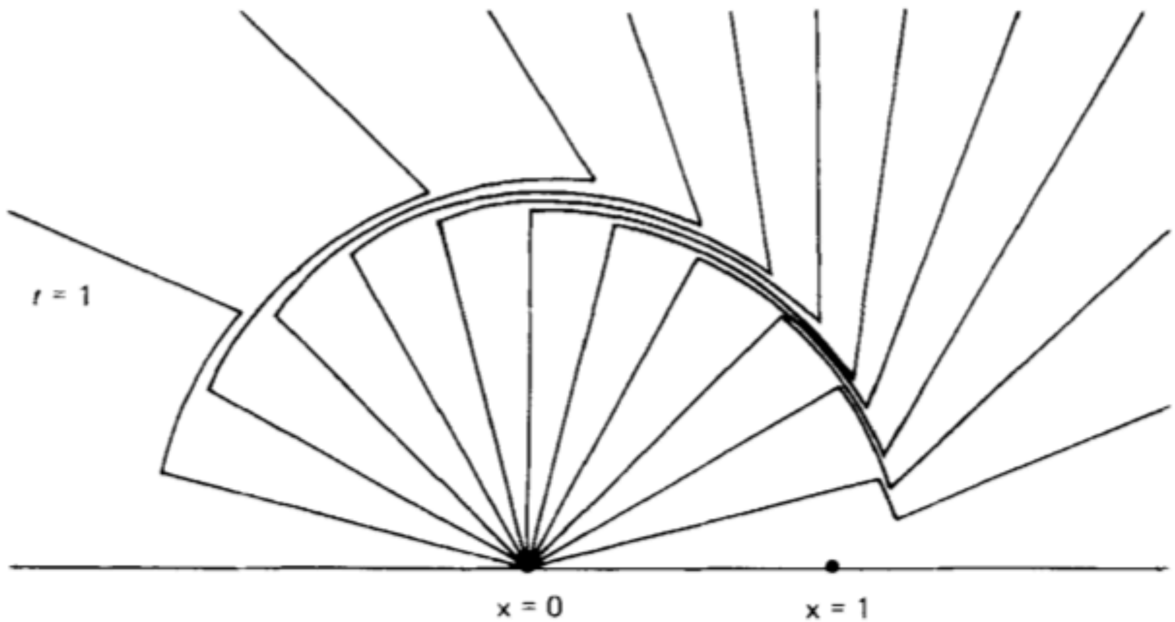
$|E_{rad}| = |B_{rad}|$

# Radiation fields

Consider a particle originally moving at constant velocity along x axis is stopped at x=0 and t=0

At t=1 the field outside of a radius c is radial and points to the position where particle would have been if there was no deceleration (since no information Is yet propagated to that distance)

But field inside the radius c is informed.




Graphical demonstration of 1/R acceleration field

# Observables

From an empiricist's point of view there are 4 observables for radiation

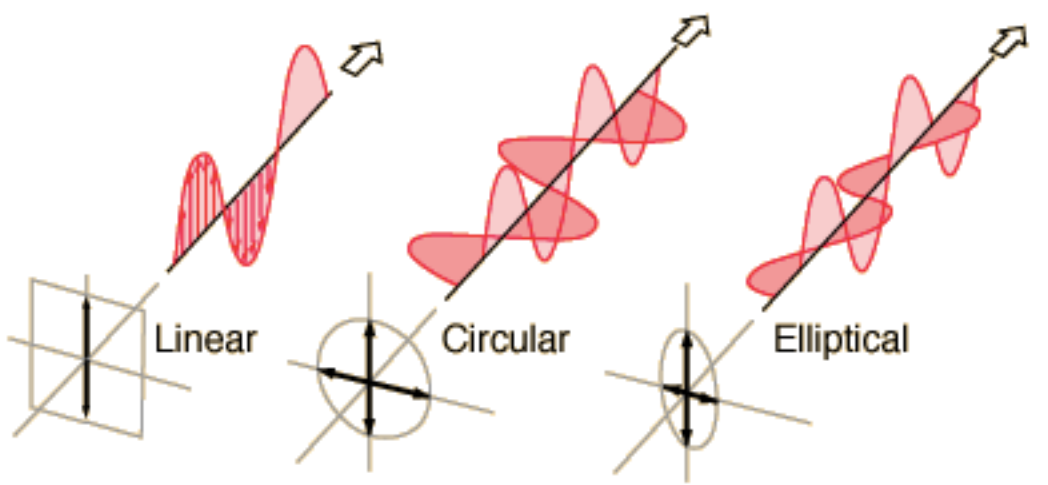
- Energy Flux
- Direction
- Frequency
- **Polarization**

Polarimetry : study of polarization of incoming radiation

# Polarization of electromagnetic radiation

- ✓ Polarization is produced in various ways, including directly from some radiation processes (e.g. cyclotron and synchrotron emission), from differential absorption of radiation passing through the interstellar medium, and perhaps most commonly from the scattering of radiation.

- ✓ Property of a wave to have its Electric Field oscillating in a single plane (plane polarized wave) or in a rotating plane (elliptically or even circular polarized wave).



# Polarization of electromagnetic radiation

- ✓ Polarization is produced in various ways, including directly from some radiation processes (e.g. cyclotron and synchrotron emission), from differential absorption of radiation passing through the interstellar medium, and perhaps most commonly from the scattering of radiation.

- ✓ Fractional polarizations detected from astronomical objects can be
  very high
  (pulsars: almost fully linearly polarised) to,
  very low
  (sun: one of the most sensitive polarization measurements ever made was by James Kemp in 1987, who showed that the fractional linear polarization of light from the Sun was ~ $10^{-7)}$

# Polarization of electromagnetic radiation

- ✓ Polarimetry, is a method used to study the polarization of incoming radiation and can provide substantial clues to the nature of the source.

- ✓ Polarimetry is used to extract information such as the strength of magnetic fields in the interstellar medium (ISM), provide evidence for inflation by observations of the CMB polarization, motivate a unified model for active galactic nuclei (AGN), probing emission geometry for pulsars etc.

# Polarization of electromagnetic radiation

- ✓ Study of polarization of electromagnetic plane waves from astrophysical sources and modification of the polarization in the medium.
- ✓ Plane waves are described by oscillating electric and magnetic fields, whose field vectors are orthogonal to each other and the direction of propagation.
- ✓ By convention, astronomers describe the polarization of light only in terms of the electric field vector (because E and B are orthogonal).

# Maximum observed or expected degree of polarization for different astronomical objects

| | |
|---|---|
| **Radio** | |
| galactic continuum | 70% |
| quasars (integrated / resolved) | 15% / 70% |
| Crab nebula | 30% |
| pulsars (linear / circular) | 80% / 70% |
| **Optical** | |
| planets | > 20% |
| interstellar dust acting on starlight (linear) | 10% |
| interstellar dust acting on starlight (circular) | 0.05% |
| Sun and $A_p$ stars (Zeeman effect) | 100% |
| white dwarfs (Zeeman effect) | 12% |
| symbiotic stars (Raman scattering) | 8% |
| reflection nebulae (including Herbig–Haro and bipolar | 60% |
| post–AGB stars and proto–PN (global polarisation) | 30% |
| synchrotron (Crab nebula, blazars) | 50% |
| synchrotron (extragalactic jets) | 20% |
| Crab pulsar | 10% |
| **X–ray (mainly 'expected')** | |
| solar flares | 5% |
| Crab nebula | 15% |
| accreting X–ray pulsars | 80% |
| rotation–powered X–ray pulsars | 10% |
| black hole (Lense–Thirring effect Cyg X–1) | 2% |
| active galactic nuclei | 20% |
| Seyfert accretion disc reprocessing | 5% |
| **$\gamma$–ray ('expected')** | |
| pulsars | 100% |

Credit: Agnieszka Słowikowska

These are approximate numbers

May not be updated

# Stokes parameters

- ✓ **The polarization can be described by the shape that the tip of E traced out over the course of a period, and it can be linear, circular, or elliptical.**

Stokes parameters were defined by George Gabriel Stokes in 1852, as a mathematically convenient alternative to the more common description of incoherent or partially polarized radiation in terms of its total intensity (I), (fractional) degree of polarization (p), and the shape parameters of the polarization ellipse

# Polarization of electromagnetic radiation

Specific case

We discussed about monochromatic plane wave

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

↓

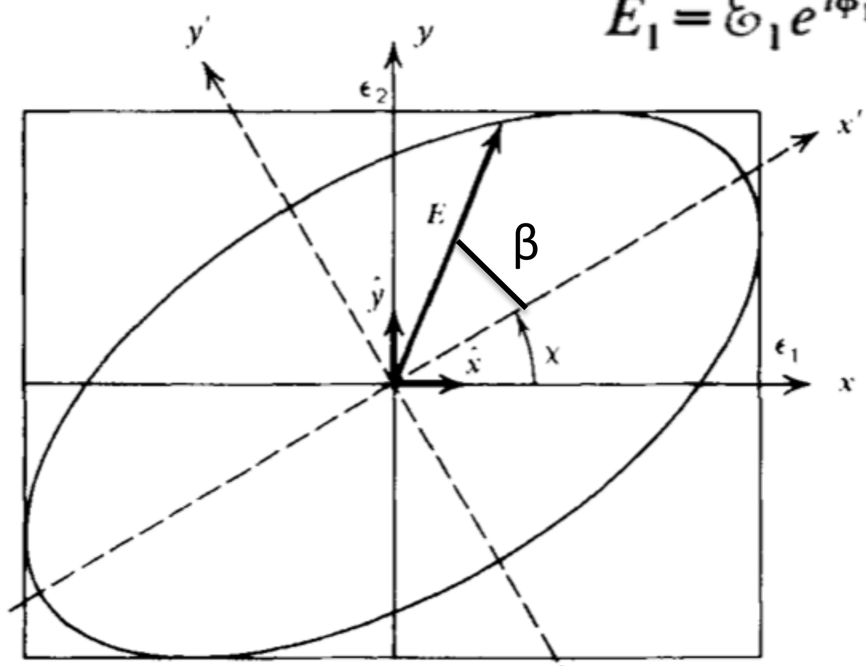
Oscillates along $a_1$

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E} = (\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2) e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}.$$

$E_1$ and $E_2$ are complex amplitude and can be written as

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2}$$



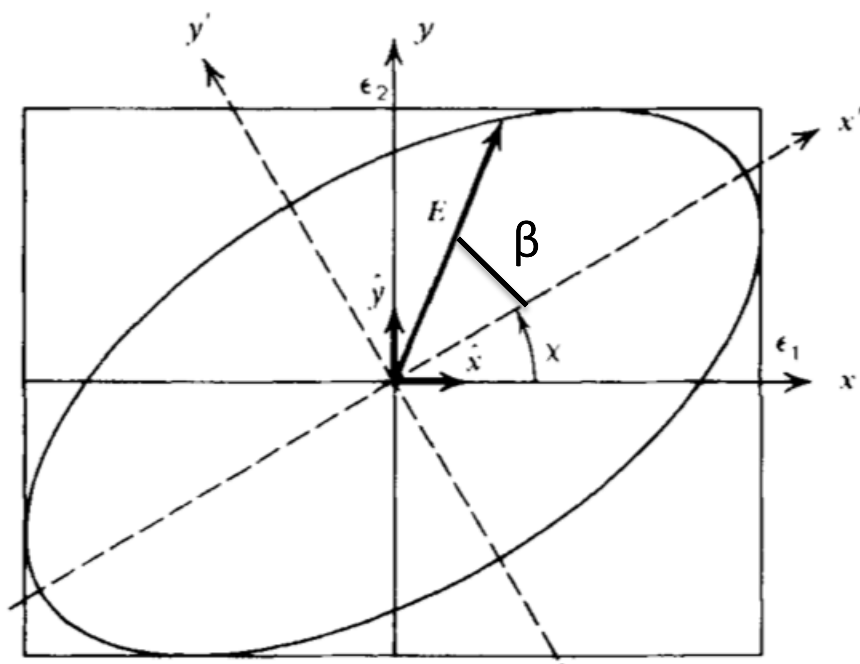
# Polarization of electromagnetic radiation

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E} = (\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2) e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}.$$

Considering real part of E, physical component of electric fields along x and y direction

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2).$$




These equations describe Tip of E in x-y plane

# Polarization of electromagnetic radiation

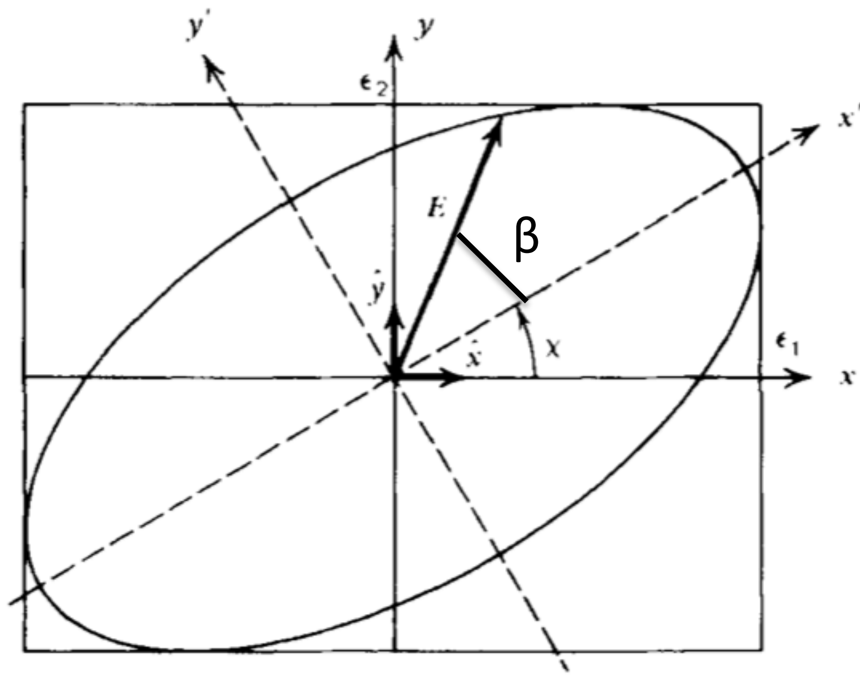
Equations describing tip of **E** in x-y plane

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2).$$

Figure traced out by tip of **E** is an ellipse

Equations for a general ellipse relative to its principal axes x' and y'

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$



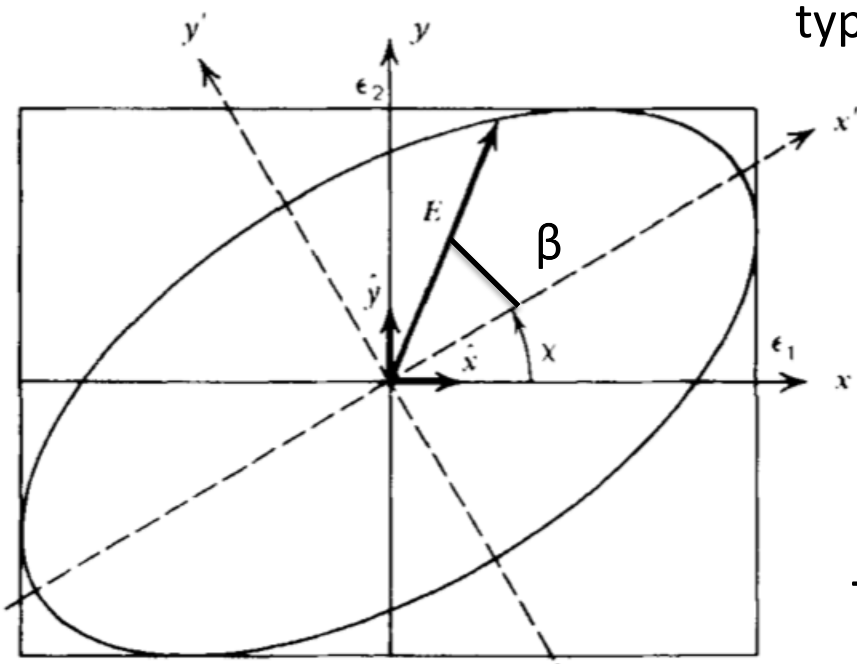
# Polarization of electromagnetic radiation

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E} = (\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2) e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}.$$

Elliptically Polarized

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$




Equation of tip of electric field vector determines type of polarisation

$$(E'_x / \mathcal{E}_0 \cos\beta)^2 + (E'_y / \mathcal{E}_0 \sin\beta)^2 = 1$$

↓

Elliptically Polarized

$0<\beta<\pi/2$ → Clockwise ellipse, Right-handed polarization

$-\pi/2<\beta<0$ → Anti-Clockwise ellipse, Left-handed polarization

# Polarization of electromagnetic radiation

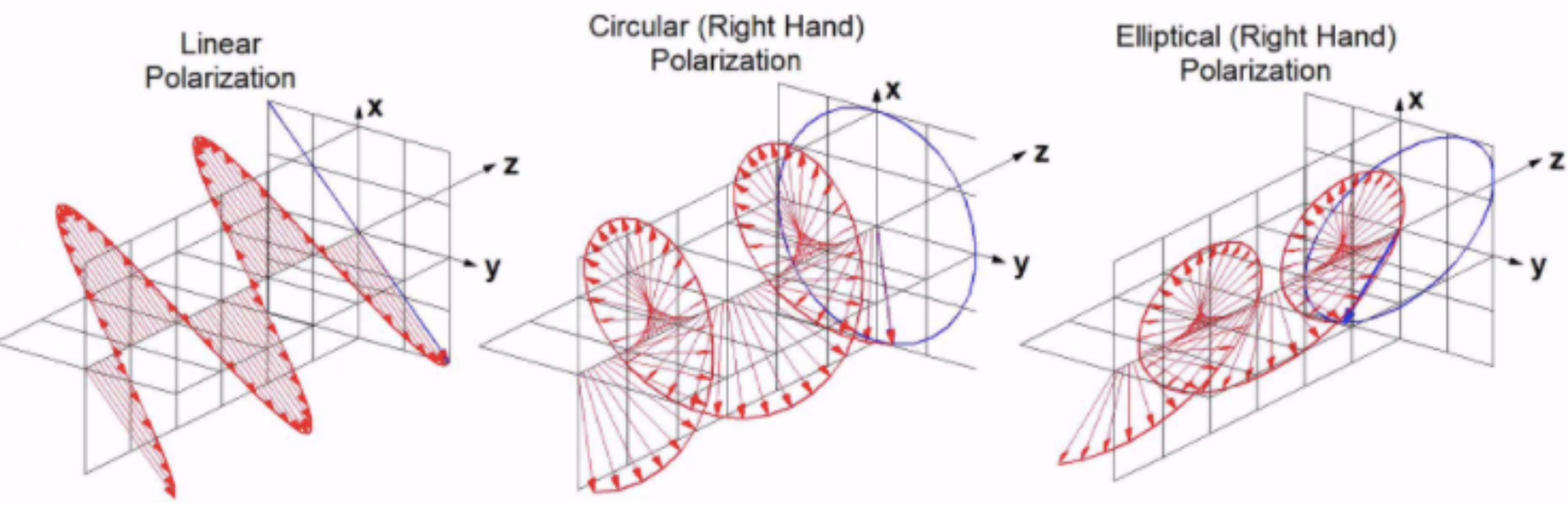
Two special cases of elliptical polarization

$\beta = \pm\pi/4$ → Circular

$\beta = 0$ , $\beta = \pi/2$ → Linear

Circular ↓

Right-handed circularly polarized
Left handed circularly polarized



# Polarization and stokes parameters

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$

Thus

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t)$$
$$E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t)$$

# Polarization and stokes parameters

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$

Thus

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t)$$

$$E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t)$$

However,

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2}$$

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2)$$

Consider,

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta\cos\chi,$$

$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta\sin\chi,$$

$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta\sin\chi,$$

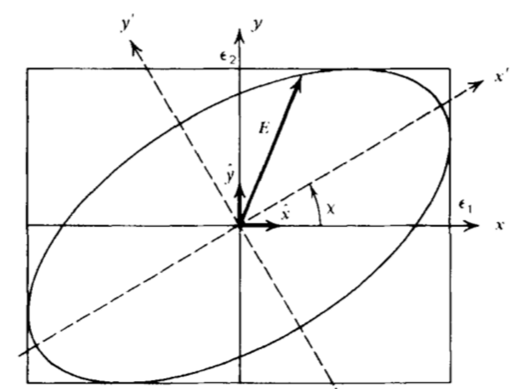
$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta\cos\chi.$$

# Polarization and stokes parameters

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi,$$
$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi,$$
$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi,$$
$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi.$$

Stokes parameters

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2$$
$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi$$
$$U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$
$$V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$



# Polarization and stokes parameters

Stokes parameters

$$I^2 = Q^2 + U^2 + V^2$$

Valid for Monochromatic wave

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2$$

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi$$

$$U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$

$$V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$

$$\mathcal{E}_0 = \sqrt{I}$$

$$\sin 2\beta = \frac{V}{I}$$

$$\tan 2\chi = \frac{U}{Q}.$$

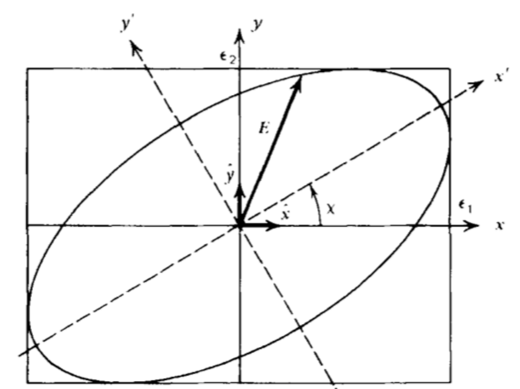
- ✓ I is Proportional to intensity of wave (+ve)

- ✓ Circularity parameter measure ratios of axes of the ellipse

+ve for Right-handed polarization
-ve for left handed polarization
V=0 for linear polarization

- ✓ Q / U measures orientation of ellipse relative to x-axis

Q=U=0 for circular polarization



# Polarization and stokes parameters

Quasi monochromatic waves, $E_1(t) = \mathcal{E}_1(t) e^{i\phi_1(t)}, \qquad E_2(t) = \mathcal{E}_2(t) e^{i\phi_2(t)}$

$$I \equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle$$

$$Q \equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \rangle$$

$$U \equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = \langle 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle$$

$$V \equiv \frac{1}{i}(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle) = \langle 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle$$

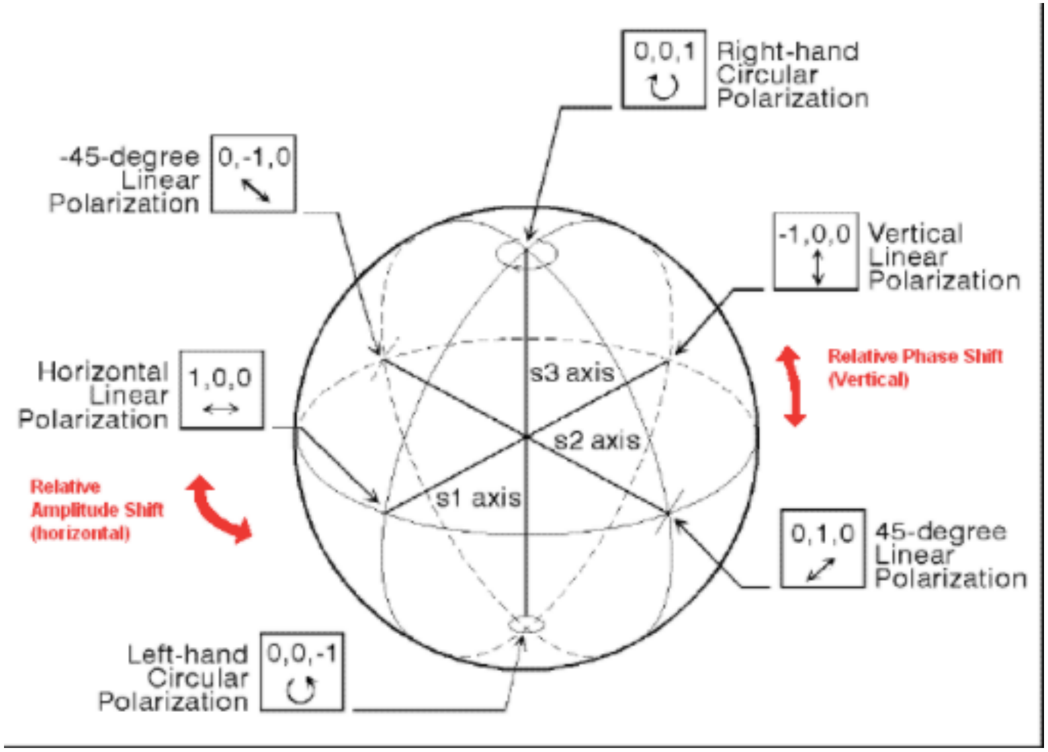
$$I^2 \geqslant Q^2 + U^2 + V^2$$

Degree of polarization,

$$\Pi \equiv \frac{I_{\text{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

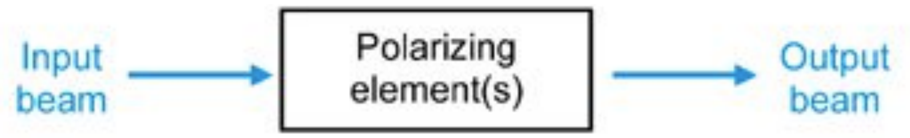
# Further reading

Poincare Sphere : a graphical tool to visualize different types of polarized radiation



# Further reading

Mueller Matrix : Method for transforming Stokes parameters




$$
\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.
$$

# End of Lecture 4

Reference: Rybicki Lightman Chapter 2,3

## Next lecture : $22^{nd}$ August

Topic of next Lecture:
Radiation from moving charges (continued)
(Chapter 3 of Rybicki & Lightman)

Preparation: Lecture 4