Electrodynamics and Radiative Processes I

Lecture 3 –

Spontaneous and stimulated emission of radiation

Problem solving on Radiative Transfer

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Lecture -2

Questions raised in the class

Sun's BB spectrum peaks at which wavelength

Deviation from BB curve fitting in real cases

How do we know components of medium from emission/absorption lines

Stimulated and spontaneous emission of radiation
Einstein Coefficients
First derivation of Planck's function

A. Einstein, Physikalische Zeitschrift **18,** 121 **1917**

The Quantum Theory of Radiation

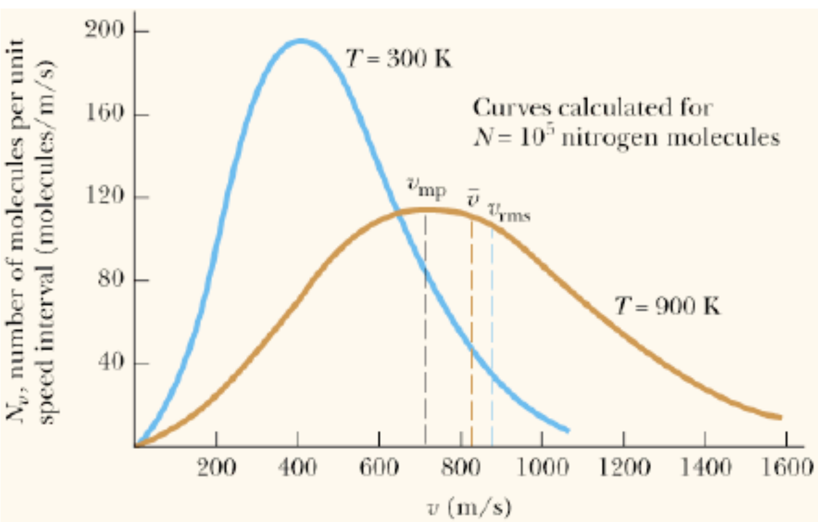
A. Einstein
(Received March, 1917)

The formal similarity of the spectral distribution curve of temperature radiation to Maxwell's velocity distribution curve is too striking to have remained hidden very long. Indeed, in the important theoretical paper in which Wien derived his displacement law

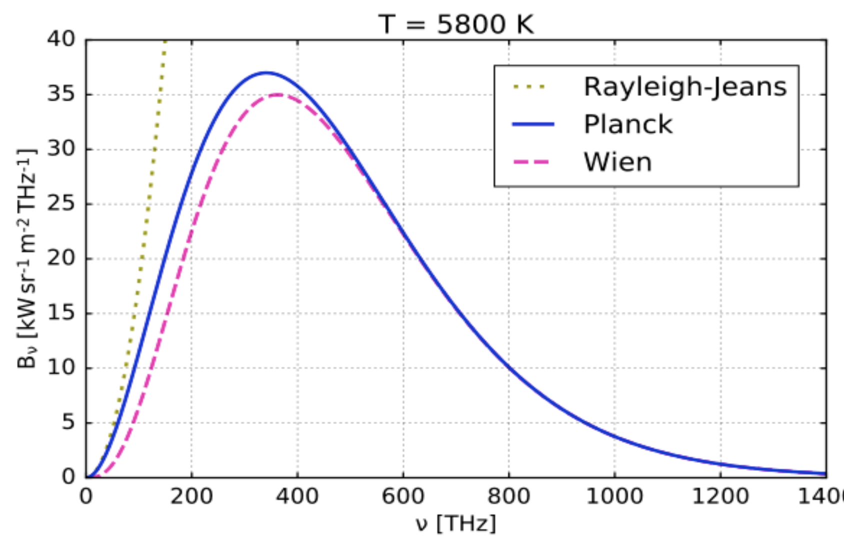
$$\rho = \nu^3 f\left(\frac{\nu}{T}\right) \tag{1}$$

"The formal similarity of the spectral distribution curve of temperature of radiation to Maxwell's velocity distribution curve is too striking to have remained hidden for very long." Einstein 2017

Maxwell's speed distribution

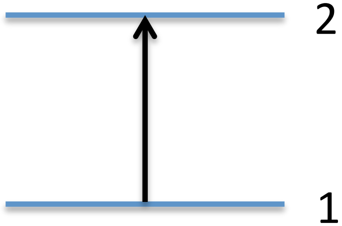


Planck's distribution



For a detailed discussion on this listen to lecture by G. Srinivasan
: https://www.youtube.com/watch?
v=o_h8djx68tw&list=PL04QVxpjcnjidFFZjZ3m0aJ6pnKfBN5sA&index=2

Einstein Coefficients



Stimulated absorption

B_{12} (dependent on radiation)

$$J_\nu = \frac{1}{4\pi}\int I_\nu \, d\Omega$$

Remember (from Lecture 1)

This occurs in presence of photons of energy $h\nu_o$

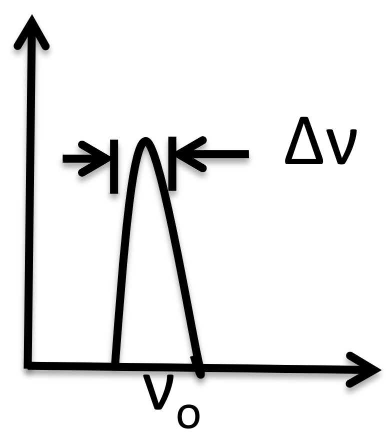
Energy difference between two levels is not infinitely sharp

Described by a line profile function $\Phi(\nu)$

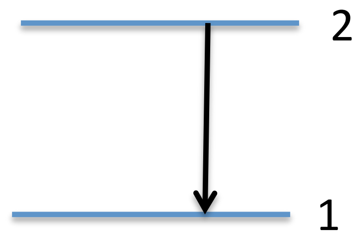
$$\int_0^\infty \phi(\nu)\, d\nu = 1$$

$$\bar{J} \equiv \int_0^\infty J_\nu \phi(\nu)\, d\nu$$

$B_{12}\bar{J}$: → Stimulated absorption rate



Einstein Coefficients



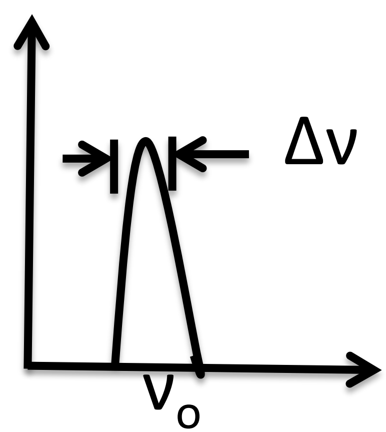
Stimulated emission

B_{21}(dependent on radiation)

Photons of energy $h\nu_o$ is emitted.

Energy difference between two levels is not infinitely sharp

Described by a line profile function $\Phi(\nu)$



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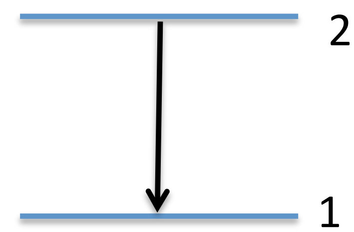
Einstein Coefficients



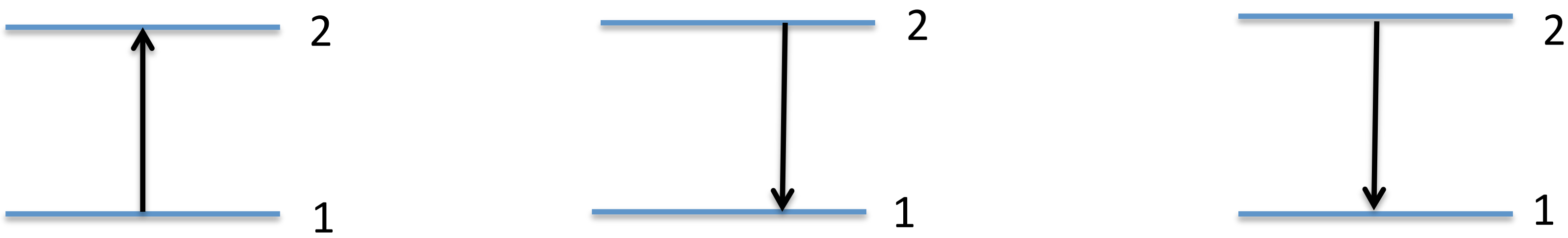
Spontaneous emission

A_{21}(independent of radiation)

Occurs when system in level 2 goes to 1, emits a photon of energy $h\nu_o$
It occurs even in absence of radiation fields

A_{21} → Spontaneous emission rate

Einstein Coefficients



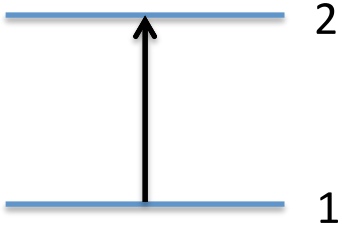
Stimulated absorption
B_{12} (dependent on radiation)

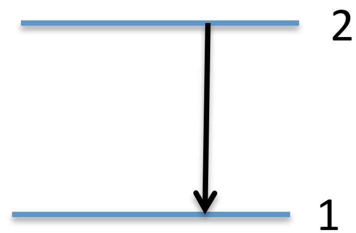
Stimulated emission
B_{21}(dependent on radiation)

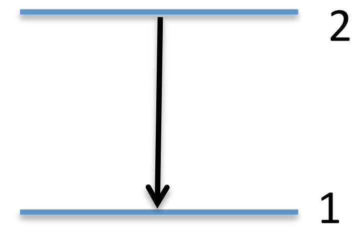
Spontaneous emission
A_{21}(independent of radiation)

Rate 1 to 2 = Rate 2 to 1 $$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}.$$

Einstein Coefficients







Stimulated absorption
B_{21} (dependent on radiation)

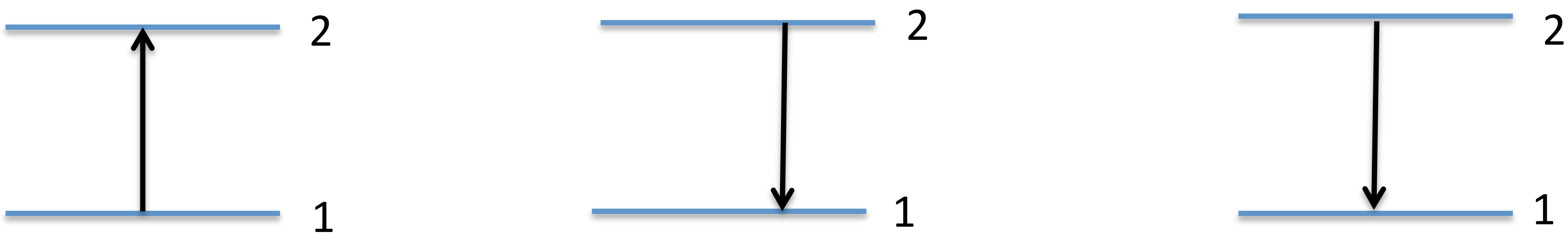
Stimulated emission
B_{12}(dependent on radiation)

Spontaneous emission
A_{21}(independent of radiation)

Rate 1 to 2 = Rate 2 to 1 $$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}.$$

In thermal equilibrium $$\frac{n_1}{n_2} = \frac{g_1 \exp(-E/kT)}{g_2 \exp[-(E+h\nu_0)/kT]} = \frac{g_1}{g_2} \exp(h\nu_0/kT)$$

Einstein Coefficients



Stimulated absorption
B_{12} (dependent on radiation)

Stimulated emission
B_{21}(dependent on radiation)

Spontaneous emission
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Mean specific intensity

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}$$ => Plank Function

Einstein Coefficients

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}$$

$$\boxed{I_\nu = B_\nu(T) = \frac{2\,h\,\nu^3/c^2}{e^{h\nu/kT} - 1}}$$

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$$g_1 B_{12} = g_2 B_{21} \qquad A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

These relations must hold whether or not there is thermodynamic equilibrium.

Einstein Coefficients

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_0/kT) - 1}$$

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$$g_1 B_{12} = g_2 B_{21}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Wien Law hν >> kT

$$I_\nu^W(T) = \frac{2h\nu^3}{c^2} \exp\left(\frac{-h\nu}{kT}\right)$$

Einstein Coefficients

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

↓

Whenever there is stimulated emission, there has to be spontaneous emission.

- Einstein coefficients connect the atomic properties A_{21} , B_{21} and B_{12} and have no relation to temperature.

- If we determine any one of these coefficient then that will allow us to determine other two.

- Einstein had to include the process of simulated emission as without it he could not get Planck's Law.

Einstein Coefficients

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Whenever there is stimulated emission, there has to be spontaneous emission.

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hv>>kT level 2 is sparsely populated compared to level 1 i.e. n2<<n1
Stimulated emission is unimportant compared to absorption, since these are proportional to n2 and n1.

Emission coefficient in terms of Einstein Coefficients

Amount of energy emitted in volume dV, solid angle dΩ frequency dν and time dt

$$dE = j_\nu \, dV \, d\Omega \, dt \, d\nu,$$ Lecture 1

Each atom contributes energy $h\nu_0$ distributed over 4π solid angle

Emission coefficient in terms of Einstein Coefficients

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$$dE = j_\nu \, dV \, d\Omega \, dt \, d\nu$$ (Lecture 1)

Each atom contributes energy $h\nu_0$ distributed over 4π solid angle

Amount of energy emitted in volume dV, solid angle dΩ frequency dν and time dt

$$dE = (h\nu_0/4\pi)\phi(\nu)n_2 A_{21} dV \, d\Omega \, d\nu \, dt$$

Emission coefficient in terms of Einstein Coefficients

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$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)$$

Absorption coefficient in terms of Einstein Coefficients

Amount of energy emitted in volume dV, solid angle dΩ frequency dν and time dt

$$dE = dV\,dt\,d\Omega\,d\nu \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) I_\nu$$

Absorption coefficient in terms of Einstein Coefficients

Amount of energy emitted in volume dV, solid angle dΩ frequency dν and time dt

$$dV\,dt\,d\Omega\,d\nu\frac{h\nu_0}{4\pi}n_1B_{12}\phi(\nu)I_\nu$$

$$\Downarrow$$

$$\alpha_\nu=\frac{h\nu}{4\pi}n_1B_{12}\phi(\nu).$$

Absorption coefficient in terms of Einstein Coefficients

Amount of energy emitted in volume dV, solid angle dΩ frequency dv and time dt

$$dV\,dt\,d\Omega\,d\nu\,\frac{h\nu_0}{4\pi}n_1B_{12}\phi(\nu)I_\nu$$

⇓

Absorption coefficient

$$\alpha_\nu=\frac{h\nu}{4\pi}n_1B_{12}\phi(\nu).$$

Absorption coefficient corrected for stimulated emission

$$\alpha_\nu=\frac{h\nu}{4\pi}\phi(\nu)(n_1B_{12}-n_2B_{21}).$$

The doubt in Lecture 1 regarding not including stimulated emission is resolved
Rybicki & Lightman problem 1.7

Radiative transfer equation in terms of Einstein Coefficients

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

Replacing the emission and the absorption coefficients in the radiative transfer equation

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21}\phi(\nu)$$

Source function

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

Radiative transfer equation in terms of Einstein Coefficients

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$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}(1 - g_1 n_2 / g_2 n_1)\phi(\nu),$$

Radiative transfer equation in terms of Einstein Coefficients

Replacing the emission and the absorption coefficients in the radiative transfer equation

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Source function $$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}(1 - g_1 n_2 / g_2 n_1)\phi(\nu),$$

$$S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1}$$

Special cases
1. Thermal emission

If the matter is in thermodynamic equilibrium

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(\frac{h\nu}{kT} \right)$$

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Correction factor due to stimulated emission

Special cases
1. Thermal emission

If the matter is in thermodynamic equilibrium

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$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \left[1 - \exp\left(\frac{-h\nu}{kT}\right)\right] \phi(\nu)$$

Correction factor due to stimulated emission

$$S_\nu = B_\nu(T)$$

Special cases
2. Non-thermal emission

For all other cases where thermal equilibrium is not achieved

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} \exp\left(\frac{h\nu}{kT}\right)$$

Special cases
3. Inverted Populations

For a system with thermal equilibrium we have

$$\frac{n_1}{g_1} > \frac{n_2}{g_2}.$$

Such systems are called normal population

It is possible to put enough atoms in the upper state so that we have population inversion

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

Absorption coefficient is negative

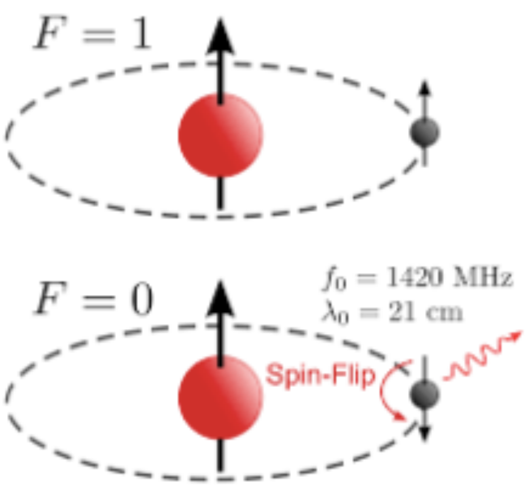
$$\alpha_\nu < 0,$$

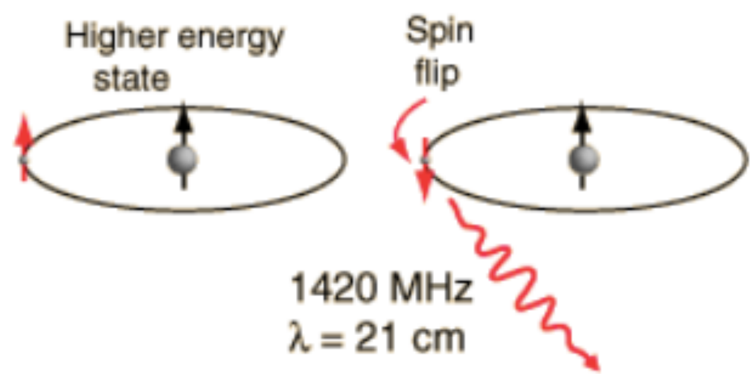
Einstein Coefficients

From a quantum electrodynamic treatment of spontaneous emission, it may be shown

$$A_{\mathrm{UL}} \approx \frac{64\pi^4}{3hc^3}\nu_{\mathrm{UL}}^3|\mu_{\mathrm{UL}}|^2$$

Radiation is due to change of dipole moment.





Further reading : https://www.cv.nrao.edu/course/astr534/HILine.html

21cm emission line

Hydrogen is the most abundant element in the interstellar medium (ISM), but the symmetric H2 molecule has no permanent dipole moment and hence does not emit a detectable spectral line at radio frequencies.

Neutral hydrogen (HI) atoms are abundant in low-density regions of the ISM. They are detectable in the 21 cm (~1420 MHz) hyperfine line.
Two energy levels result from the magnetic interaction between quantized electron and proton spins. When the relative spins change from parallel to antiparallel, a photon is emitted.

$$A_{10} \approx 2.85 \times 10^{-15}\ \mathrm{s}^{-1}$$

$$\tau_{1/2} = A_{10}^{-1} \approx 3.5 \times 10^{14}\ \mathrm{s} \approx 11 \text{ million years}$$

However a large fraction of what we know about the universe comes from studying the universe at 21 cm

Problem solving

On Radiative Transfer

Radiative transfer
(Lecture 1-3)

$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

$$dI_\nu = j_\nu \, ds \longrightarrow I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s') \, ds'$$

$$dI_\nu = -\alpha_\nu I_\nu \, ds \longrightarrow I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s') \, ds' \right]$$

$$\alpha_\nu = n\sigma_\nu$$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s') \, ds'$$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \longrightarrow \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$$

Radiative transfer

Low frequency regime

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

$$\frac{dT_b}{d\tau_\nu} = -T_b + T, \quad \longrightarrow \quad T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu})$$

Einstein's coefficients

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \quad \longrightarrow \quad g_1 B_{12} = g_2 B_{21} \qquad A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Problem solving
On Radiative Transfer

Type 1: Calculate luminosity, flux density , specific intensity, specific energy density, brightness temperature (example 1.1,1.3,1.5 of Rybicki & Lightman)

Type 2: Optically thick/thin medium (example 1.8 of Rybicki & Lightman)

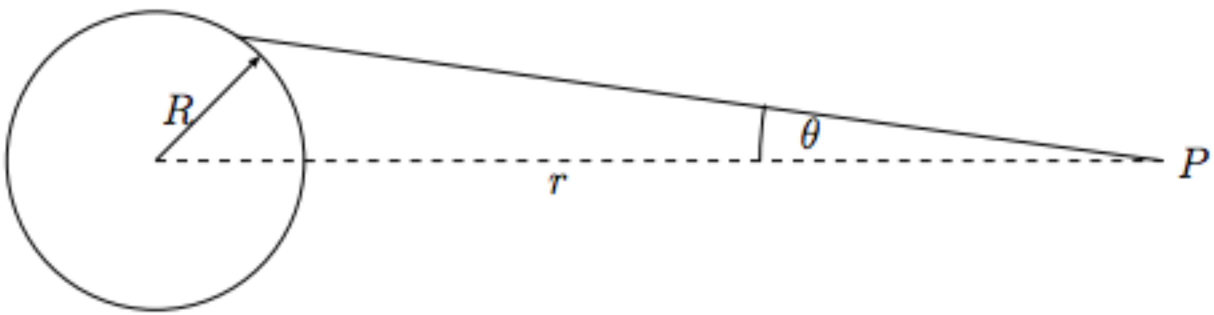
Type 3: Emission and absorption lines (example 1.9 of Rybicki & Lightman)

Type 4: Miscellaneous (example 1.7, 1.10 of Rybicki & Lightman)

Example 1: Calculate the total flux at a point P coming from an isotropic, optically thick sphere of radius R.

Optically thick: emission comes from only surface

Isotropic: I is same for each point of the surface



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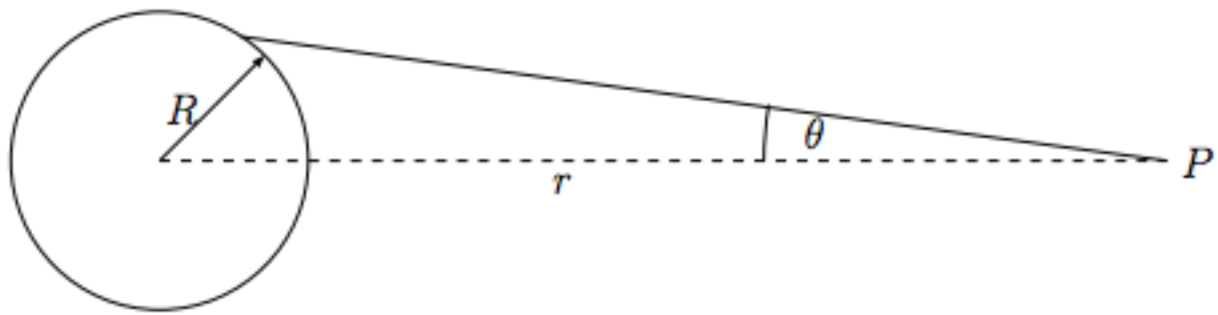

Solution

$$F_\nu = I_\nu \int_0^{\theta_c} \int_0^{2\pi} \cos\theta \sin\theta \, d\phi d\theta, \quad = \pi I_\nu \sin^2 \theta_c,$$

$$F_\nu = \pi I_\nu \left(\frac{R}{r}\right)^2$$

Example 2: Upon reaching a planet, part of the central star radiation will be reflected and the rest will be absorbed and re-emitted as a cooler blackbody. In the spectra of planets both of these components are observed. Consider the case of Jupiter, with radius $R_J = 7.1 \times 10^9$ cm and mean orbital radius $a_J = 7.8 \times 10^{13}$ cm. Assume that the spectrum of the Sun is a perfect blackbody.

(a) Suppose that Jupiter perfectly reflects 10% of the light coming from the Sun. Calculate its reflected luminosity. At which wavelength does it peak? In which spectral band is it observed?

(b) At which wavelength does the re-emitted luminosity peaks? In which spectral band is it observed?

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Hint for Solution:

(a) Solar flux reaching Jupiter $\frac{L_\odot}{4\pi a_J^2}$

Reflected Luminosity $$L_{\rm refl} = 0.1 \times \frac{L_\odot}{4}\left(\frac{R_J}{a_J}\right)^2$$

(b) For reflected emission
Find wave length from Wien's law $$4\pi\sigma T_{\rm eff}^4 R_J^2 = 0.9 \times \frac{L_\odot}{4}\left(\frac{R_J}{a_J}\right)^2$$

End of Lecture 3

Reference: Rybicki and Lightman Chapter 1.6

Lecture by G. Srinivasan : https://www.youtube.com/watch?v=o_h8djx68tw&list=PL04QVxpjcnjidFFZjZ3m0aJ6pnKfBN5sA&index=2

Next lecture : 19th August

Topic of next Lecture:
Basic theory of radiation fields
(Chapter 2 of Rybicki & Lightman)

Preparation: 2.1 and 2.2 of Rybicki & Lightman
Read about Maxwell's equations