Electrodynamics and Radiative Processes I

Lecture 11 – Synchrotron III +Problem solving

Bhaswati Bhattacharyya
bhaswati@ncra.tifr.res.in

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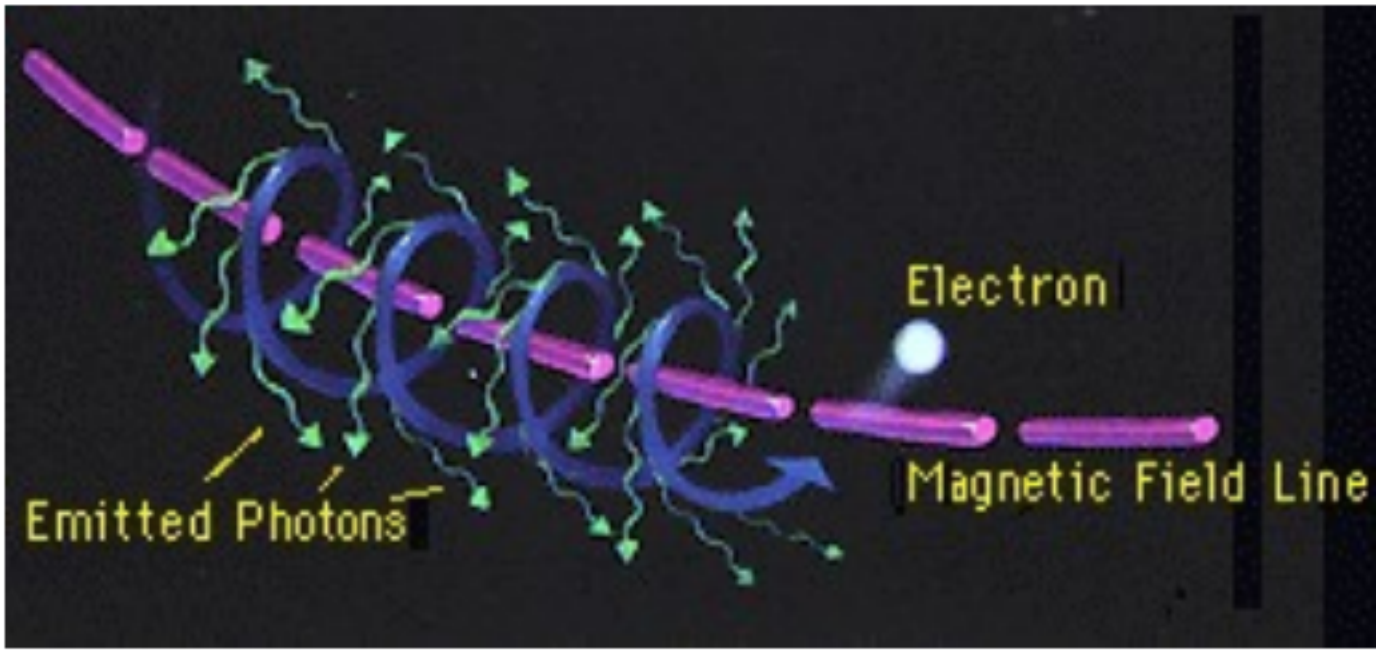
IUCAA-NCRA Graduate School

Reference :
1) Rybicki and Lightman
2) Ghisellini: http://www.brera.inaf.it/utenti/gabriele/total.pdf

Date : 12th September 2019

Synchrotron Radiation(Recap)

Synchrotron Radiation is radiation from a charge moving relativistically that is accelerated by a magnetic field.



Frequency of Gyration → $\nu_B = \omega_B / 2\pi = qB/2\pi m\gamma$

Frequency of Cut-off → $\nu_c = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha,$

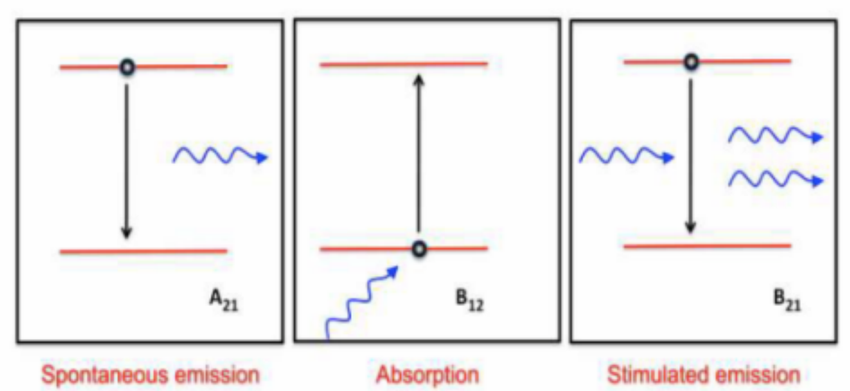
Synchrotron self-absorption

Synchrotron emission process is accompanied by absorption in which

a) A photon interacts with a charge in magnetic field and is absorbed giving up its energy to the charge

b) Stimulated emission (or negative absorption) in which a particle is induced to emit more strongly into a direction and at a frequency where photons are already present

These processes are related by Einstein's coefficients



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$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

$\varphi_{21}(\nu)$ is δ function that restricts summations to these states differing by an energy $h\nu = E_2 - E_1$

Synchrotron self-absorption

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

Now we want to write the absorption coefficient so that it contain the expression of power which we discussed,

$$\boxed{P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F\left(\frac{\omega}{\omega_c}\right)}$$

It is convenient to write the emission in terms of the frequency ν rather than ω. So we use $P(\nu, E_2) = 2\pi P(\omega)$.

Synchrotron self-absorption

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Relations between Einstein's coefficients

$$g_1 B_{12} = g_2 B_{21},$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Total power emitted per frequency of a single particle can be written as

$$\begin{aligned} P(\nu, E_2) &= h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) \\ &= (2h\nu^3/c^2) h\nu \sum_{E_1} B_{21} \phi_{21}(\nu) \end{aligned}$$

This expression relates the spontaneous emission (A21) with the stimulated emission (B21)

Synchrotron self-absorption

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

Absorption coefficient due to stimulated emission

$$\frac{-h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_2) B_{21} \phi_{21} = \frac{-c^2}{8\pi h \nu^3} \sum_{E_2} n(E_2) P(\nu, E_2).$$

Absorption coefficient due to true absorption

$$\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} n(E_1) B_{12} \phi_{21} = \frac{c^2}{8\pi h \nu^3} \sum_{E_2} n(E_2 - h\nu) P(\nu, E_2)$$

$$\alpha_\nu = \frac{c^2}{8\pi h \nu^3} \sum_{E_2} \left[n(E_2 - h\nu) - n(E_2) \right] P(\nu, E_2)$$

Synchrotron self-absorption

Consider isotropic electron distribution function f(p)

f(p) d^3p =number of electrons per unit volume with momentum in d^3p about p

$$= \quad \tilde{\omega} h^{-3} d^3 p,$$

(statistical weight of the particle, it has nothing to do with angular frequency; for electrons it's 2 (spin up/spin down states))

So we can make the substitution $$\sum_2 \rightarrow \frac{\tilde{\omega}}{h^3} \int d^3 p_2, \qquad n(E_2) \rightarrow \frac{h^3}{\tilde{\omega}} f(p_2).$$

Synchrotron self-absorption(Recap)

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\sum_{E_2}\left[n(E_2-h\nu)-n(E_2)\right]P(\nu,E_2) \quad + \quad \sum_2 \rightarrow \frac{\tilde{\omega}}{h^3}\int d^3p_2, \qquad n(E_2)\rightarrow \frac{h^3}{\tilde{\omega}}f(p_2)$$

So the absorption coefficient is

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_2^*)-f(p_2)\right]P(\nu,E_2)$$

where p_2^* is the momentum corresponding to energy E_2-hν

Check if this formula produces correct results for thermal distribution of particles

$$f(p) = K\exp\left[-\frac{E(p)}{kT}\right].$$

Synchrotron self-absorption (Recap)

$$f(p_2^*) - f(p_2) = K\exp\left(-\frac{E_2 - h\nu}{kT}\right) - K\exp\left(-\frac{E_2}{kT}\right)$$

$$= f(p_2)(e^{h\nu/kT} - 1).$$

The absorption coefficient is

$$(\alpha_\nu)_{\text{thermal}} = \underbrace{\frac{c^2}{8\pi h\nu^3}(e^{h\nu/kT} - 1)}_{1/4\pi B_\nu(T)} \underbrace{\int d^3p_2 f(p_2) P(\nu, E_2)}_{\text{total power per unit volume per unit frequency range } 4\pi j_\nu}$$

$$\boxed{(\alpha_\nu)_{\text{thermal}} = \frac{j_\nu}{B_\nu(T)}}$$

Correct result for thermal emission
Kirchhoff's Law

Synchrotron self-absorption

The next step is to consider the power law distribution of particles and get rid of f(p)

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int d^3p_2\left[f(p_2^*)-f(p_2)\right]P(\nu,E_2)$$

$$\Big\downarrow \quad N(E)dE = f(p)4\pi p^2 dp$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int dE\, P(\nu,E)E^2\left[\frac{N(E-h\nu)}{(E-h\nu)^2}-\frac{N(E)}{E^2}\right]$$

(replaced E_2 by E)

$$\Big\downarrow \quad h\nu \ll E$$

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE\, P(\nu,E)E^2\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]$$

Synchrotron self-absorption

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE\, P(\nu,E)E^2\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right].$$

$$\Big\downarrow \quad N(E) = KE^2 e^{-E/kT}.$$

$$(\alpha_\nu)_{\text{thermal}} = \frac{c^2}{8\pi\nu^2 kT}\int N(E)P(\nu,E)\,dE = \frac{j_\nu c^2}{2\nu^2 kT}$$

This is Kirchhoff's law in Rayleigh-Jeans Regime (expected as hν<< E)

Synchrotron self-absorption

The last step is to consider the power law distribution of particles and get rid of f(p)

$$N(E)dE = CE^{-p}dE$$

Do some algebra (Following R&L Sect 6.8)

$$\alpha_\nu = \frac{\sqrt{3}\, q^3}{8\pi m}\left(\frac{3q}{2\pi m^3 c^5}\right)^{p/2} C(B\sin\alpha)^{(p+2)/2}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right)\nu^{-(p+4)/2}.$$

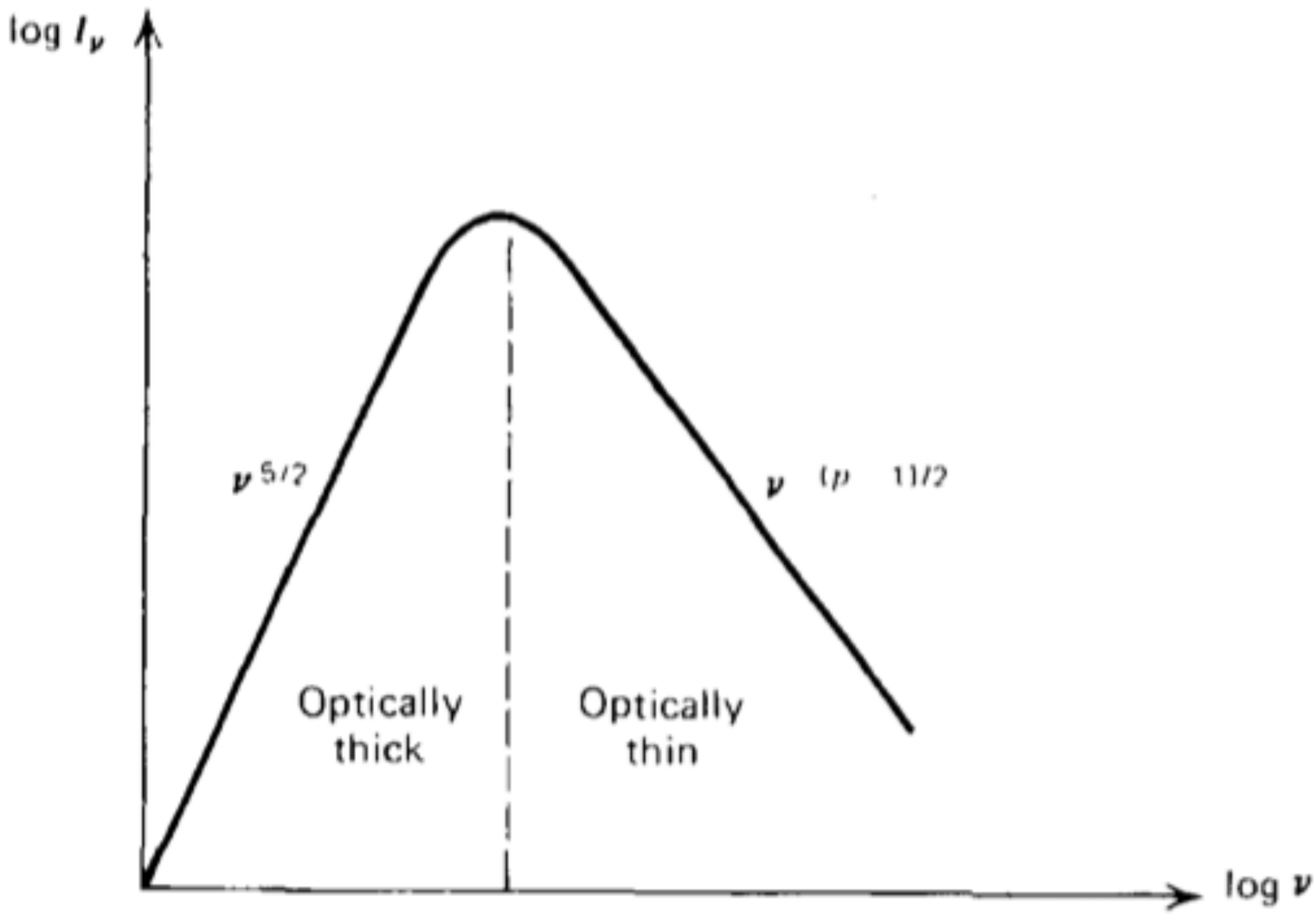
$$\boxed{S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{P(\nu)}{4\pi\alpha_\nu} \propto \nu^{5/2},}$$

Independent of p

Note that the slope is not 2 as in the Rayleigh-Jeans regime, but it's 5/2.

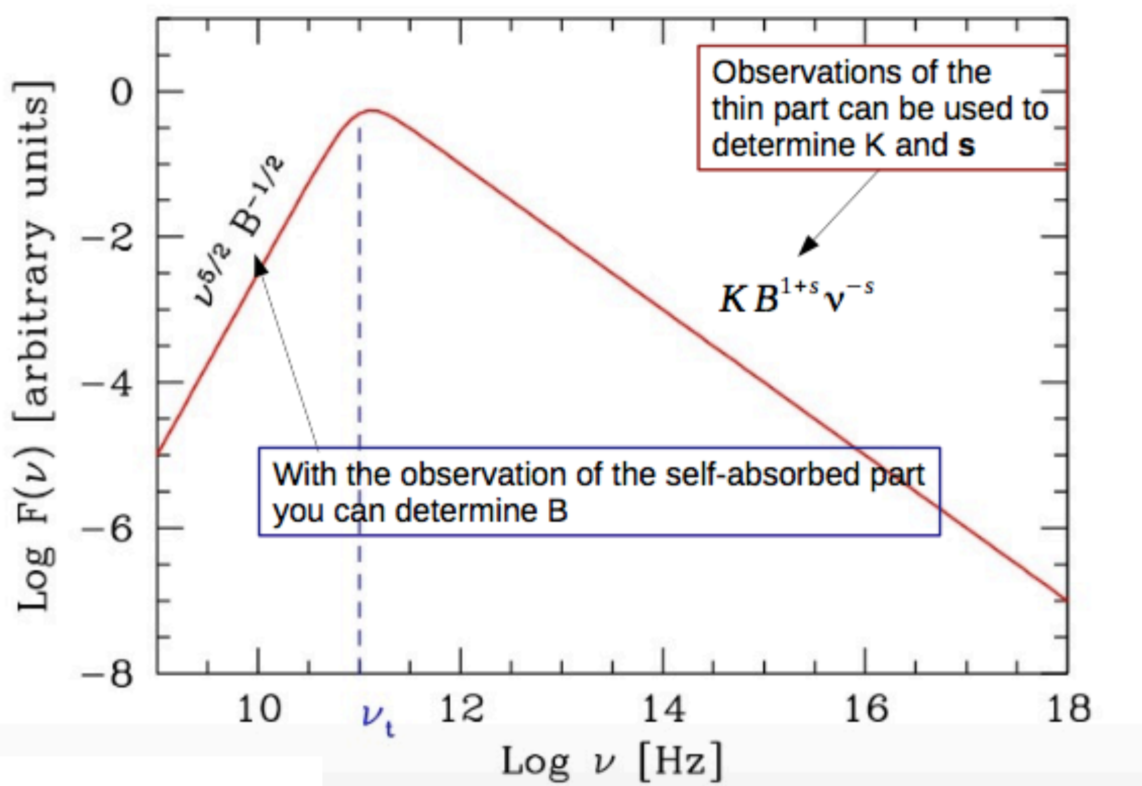
Here we do not have (and cannot have) a thermal distribution of particles emission is non thermal

Synchrotron self-absorption



Synchrotron spectrum from power-law distribution of electrons

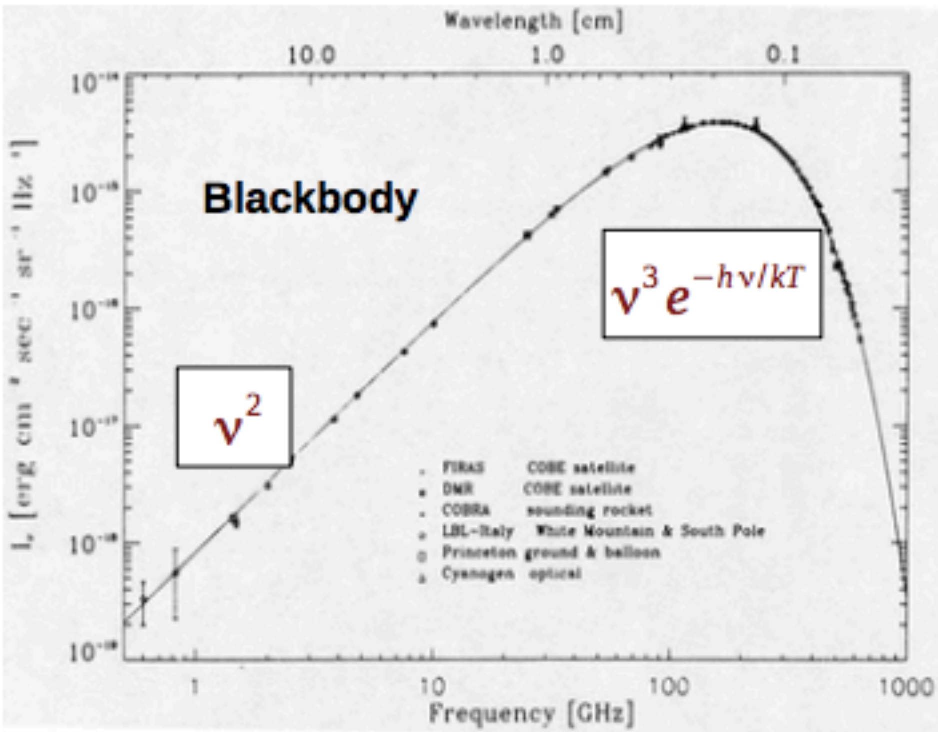
Synchrotron self-absorption

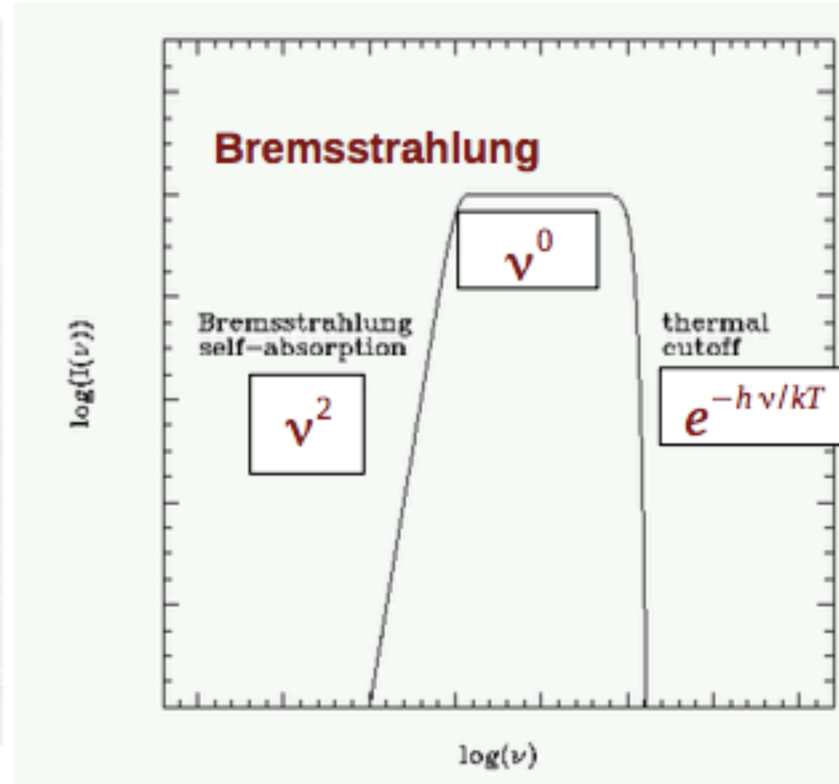


$$\nu_t \propto \left[RK\, B^{(p+2)/2} \right]^{2/(p+4)}$$

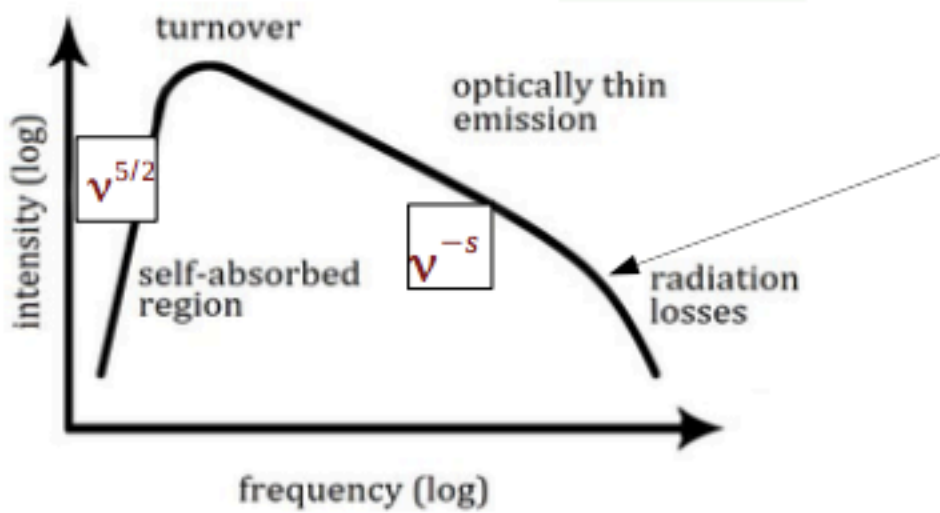
Self absorption frequency: marks the transition from optically thin to thick

Ghisellini: http://www.brera.inaf.it/utenti/gabriele/total.pdf





Synchrotron



What happens here?

Radiative processes at a glance
+ few problems from each topic

Radiative transfer

We can measure the following quantities:

- ✓ The energy in the radiation as a function of

a) position on the sky

b) frequency

- ✓ The radiation's polarisation

From these measurements we can hope to determine

- ✓ Physical parameters of the source (e.g. temperature, composition, size)
- ✓ The radiation mechanism
- ✓ The physical state of the matter

Need to understand the difference between often used terms: luminosity, flux density, specific intensity and specific energy density.

Radiative transfer
(Lecture 1-3)

$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

$$dI_\nu = j_\nu \, ds \longrightarrow I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\, ds'$$

$$dI_\nu = -\alpha_\nu I_\nu \, ds \longrightarrow I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s')\, ds' \right]$$

$$\alpha_\nu = n\sigma_\nu$$

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\, ds'$$

$$\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu} \longrightarrow \boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu}$$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

Radiative transfer

Low frequency regime

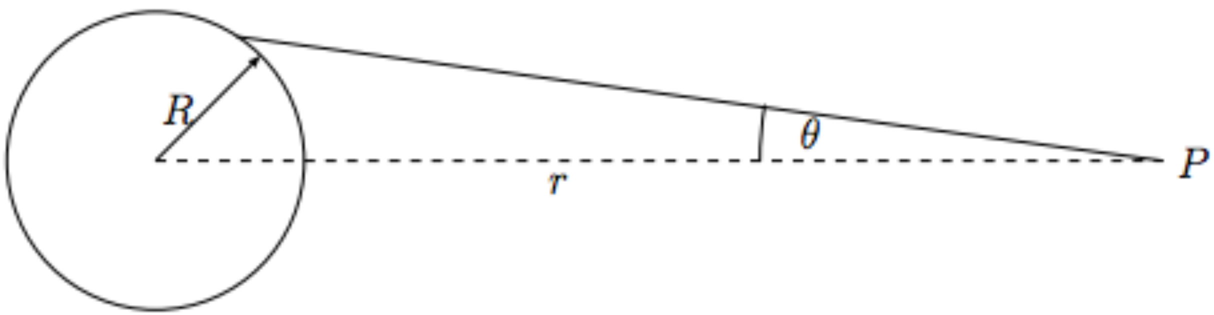
$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

$$\frac{dT_b}{d\tau_\nu} = -T_b + T, \longrightarrow T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu})$$

Einstein's coefficients

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \longrightarrow g_1 B_{12} = g_2 B_{21} \qquad A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

Example 1: Calculate the total flux at a point P coming from an isotropic, optically thick sphere of radius R.
Optically thick: emission comes from only surface
Isotropic: I is same for each point of the surface



Example 2: Upon reaching a planet, part of the central star radiation will be reflected and the rest will be absorbed and re-emitted as a cooler blackbody. In the spectra of planets both of these components are observed. Consider the case of Jupiter, with radius $R_J = 7.1 \times 10^9$ cm and mean orbital radius $a_J = 7.8 \times 10^{13}$ cm. Assume that the spectrum of the Sun is a perfect blackbody.

(a) Suppose that Jupiter perfectly reflects 10% of the light coming from the Sun. Calculate its reflected luminosity. At which wavelength does it peak? In which spectral band is it observed?

(b) At which wavelength does the re-emitted luminosity peaks? In which spectral band is it observed?

Basic Theory of Radiation Fields
(Lecture-4)

Maxwell's equation

$$\nabla \cdot \mathbf{D} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Poynting's theorem

$$\mathbf{j}\cdot\mathbf{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = -\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)$$

Radiation Spectrum

$$\Delta\omega\Delta t > 1$$

Degree of polarization

$$\Pi = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

Polarization and stokes parameter

Monochromatic light

$$\boxed{I^2 = Q^2 + U^2 + V^2}$$

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2$$

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi$$

$$U \equiv 2\mathcal{E}_1\mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$

$$V \equiv 2\mathcal{E}_1\mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$

Dipole approximation
(Lecture-5)

Differences in retarded time across source is negligible $\Longleftrightarrow$ $\lambda \gg L$,

$$\mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0} \Longleftrightarrow \frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta, \Longleftrightarrow \boxed{P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}}$$

Spectrum of radiation for dipole approximation $\quad \frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\hat{d}(\omega)|^2$

$$\boxed{\mathbf{d} = -\left(\frac{e^2 E_0}{m\omega_0^2}\right) \boldsymbol{\epsilon} \sin \omega_0 t,}$$

Thomson scattering cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}} = \frac{e^4}{m^2 c^4} \sin^2\Theta = r_0^2 \sin^2\Theta$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1-\mu^2) d\mu. = \frac{8\pi}{3} r_0^2 \quad = \sigma_T \sim 0.66 \times 10^{-24}\ \text{cm}^2$$

Problem solving

Example 1 An optically thin cloud surrounding a luminous object is estimated to be 1pc in radius. If the central object is clearly seen, what is an upper bound for the electron density of the cloud, assuming that the cloud is homogeneous.

Problem solving

Example 2: Consider a particle of mass m and charge e moving ($v \ll c$) in a constant magnetic field B. Show that the frequency of circular motion is $\omega_B = eB/mc$. Find out total emitted power. (RL 3.2)

Relativity in Electrodynamics
(Lecture 6)

Special theory of relativity

Length contraction (length of a moving rod appears smaller) $L=\left(1-\frac{v^2}{c^2}\right)^{1/2}L_0$

Time dilation (moving clock appears slower) $T=t_2-t_1=\gamma(t_2'-t_1')=\gamma T_0$

Transformation of velocities $u_{\parallel}=\frac{u_{\parallel}'+v}{\left(1+vu_{\parallel}'/c^2\right)}, \qquad u_{\perp}=\frac{u'_{\perp}}{\gamma\left(1+vu_{\parallel}'/c^2\right)}$

Addition of velocities

Beaming effect $\theta\sim\frac{1}{\gamma}$

Energy of a moving body $E_k=m_oc^2/\sqrt{(1+v^2/c^2)}$

Relativistic Doppler effect $\omega=\frac{2\pi}{\Delta t_A}=\frac{\omega'}{\gamma\left(1-\frac{v}{c}\cos\theta\right)}$

Proper time $c^2d\tau^2=c^2dt^2-\left(dx^2+dy^2+dz^2\right)$

Four vectors

Space-time is a four-vector: $x^\mu = [ct, x]$
For $\mu=0,1,2,3$

Problems based on these relations

Relativity in Electrodynamics (Lecture 6)

Examples of four vectors

Space-time is a four-vector: $x^\mu = [ct, \mathbf{x}]$

Four-vectors have Lorentz transformations between two frames with uniform relative velocity v:

$$x' = \gamma(x - \beta ct); \qquad ct' = \gamma(ct - \beta x)$$

$$x^\mu x^\nu = c^2t^2 - |x|^2 = c^2t'^2 - |x'|^2 = s^2$$

Charge/current four-vector $J^\mu = [c\rho, \mathbf{J}]$

Potential four-vector $A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$

Continuity equation, Lorentz gauge condition, Poisson's equations in terms of four vectors

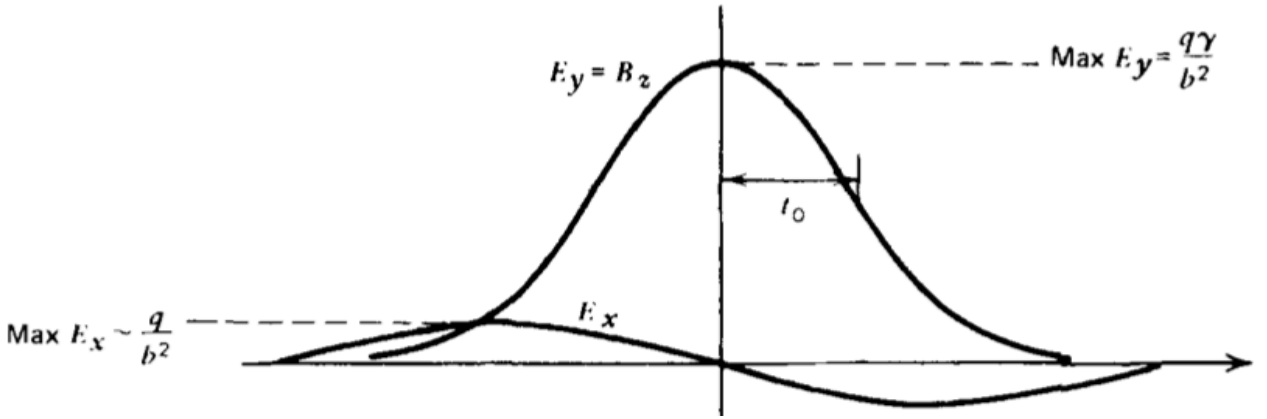
Electromagnetic field tensor

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

Express Maxwell's equations in terms of electromagnetic field tensor

Relativity in Electrodynamics (Lecture 6)

Method of virtual quanta



Fields are mostly transverse (in y direction) since (Max E_x)/(Max E_y)= Y

Fields of the moving charges are concentrated in the plane transverse to its motion into an angle of order of 1/γ

$$\boxed{P = \frac{dW}{dt}, \qquad P' = \frac{dW'}{dt'}}$$

Emitted power is Lorentz invariant

$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a_\parallel'^2 + a_\perp'^2\right) = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

Relativity

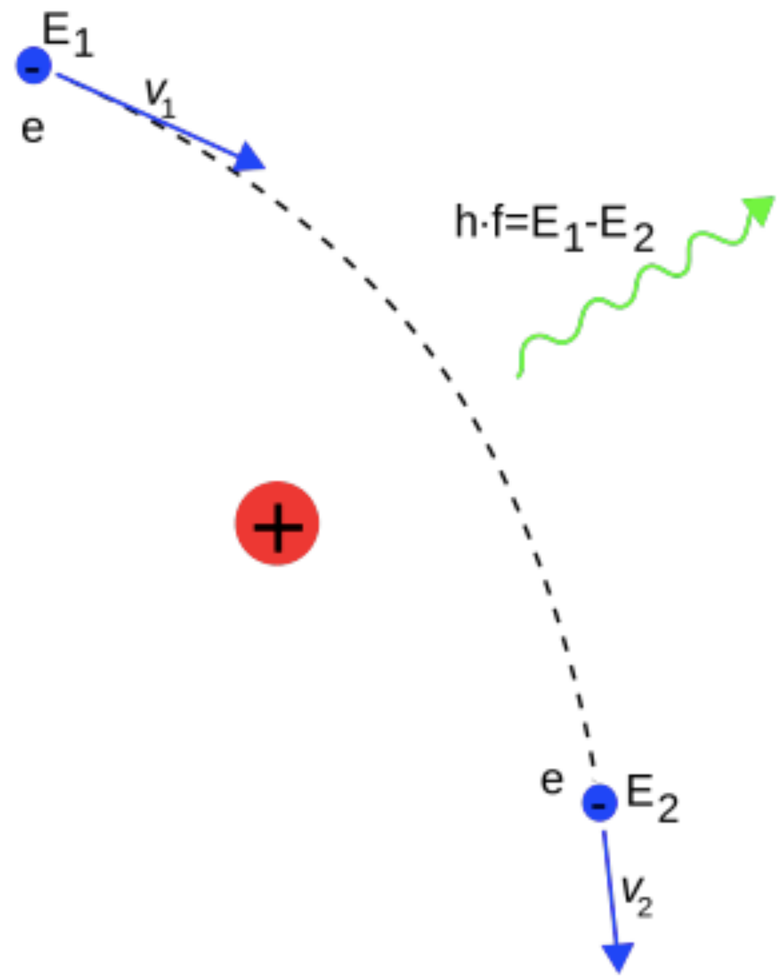
Example 1: A source has a specific intensity I_ν and is moving relativistically with respect to a stationary observer (in the reference frame K). Derive the relationship between the specific intensity measured by an observer in the reference frame K and the specific intensity seen in the comoving frame K_0 (i.e., the frame where the source is at rest).

Bremsstrahlung
(Lecture 7,8)

✓ Electrons in a plasma are accelerated by encounters with massive ions.

✓ This is the dominant continuum emission mechanism in thermal plasmas.

✓ An important *coolant* for plasmas at high temperature.



Bremsstrahlung
(Lecture 7,8)

Emissivity (energy emitted per unit volume per unit time per unit frequency)

$$\boxed{\frac{dW}{dV\,dt\,d\nu} = \frac{2^5\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}}$$

$$\boxed{\epsilon_\nu^{ff} \equiv \frac{dW}{dV\,dt\,d\nu} = 6.8\times10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}}$$

$$\boxed{\epsilon^{ff} \equiv \frac{dW}{dt\,dV} = 1.4\times10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B}$$

Free-free absorption coefficient

$$\alpha_\nu^{ff} = \frac{4e^6}{3mhc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}\left(1-e^{-h\nu/kT}\right)\bar{g}_{ff}$$

$$\alpha_\nu^{ff} = 3.7\times10^{8} T^{-1/2} Z^2 n_e n_i \nu^{-3}\left(1-e^{-h\nu/kT}\right)\bar{g}_{ff}$$

Bremsstrahlung
(Lecture 7,8)

- ✓ At low ν, $\tau_\nu >> 1$ → Black body like spectrum
- ✓ At high ν, $\tau_\nu << 1$ → Flat spectrum

Turn over at $h\nu = kT$

Non thermal Bremsstrahlung

$$\frac{dW}{dV\,dt} = 1.4 \times 10^{-27} T^{1/2} Z^2 n_e n_i \bar{g}_B (1 + 4.4 \times 10^{-10} T)$$

Problem solving
(bremsstrahlung)

Example 1: A ionized hydrogen gas ($Z = 1$) of a number density $ne = ni = 6 \times 10^2$ cm^{-3} of a size 10^{19} cm and initial temperature of 8000K cools via thermal bremsstrahlung.

(a) How long does it take for the gas to cool down to $T = 0$? Assume Gaunt factor $g = 1$.

(b) Find the luminosity of the entire nebula in terms of solar luminosities.

Problem solving
(bremsstrahlung)

Example 2: Orion nebula is one of the the brightest HII regions on the sky. Its angular size is approximately 1 deg and we know that its distance is around 400 pc. This region emits thermal bremsstrahlung radiation with transition from optically thick to optically thin regime at 1 GHZ and cut-off frequency at 200 THz.

(a) In observations, the measure of the cutoff frequency is a way to determine the plasma temperature. Estimate the temperature of NGC1976.

(b) Assuming that NGC1976 has a spherical shape, estimate the number density of the region

Problem solving
(bremsstrahlung)

Example 3: Consider a sphere of ionized hydrogen plasma that is undergoing spherical gravitational collapse. The sphere is held at constant isothermal temperature T_0, uniform density and constant mass M_0 during the collapse, and has decreasing radius R(t). The sphere cools by emission of bremsstrahlung radiation in its interior. At $t = t_0$ the sphere is optically thin.

(a) What is the total luminosity of the sphere as a function of M_0, R(t) and T_0 while the sphere is optically thin ?

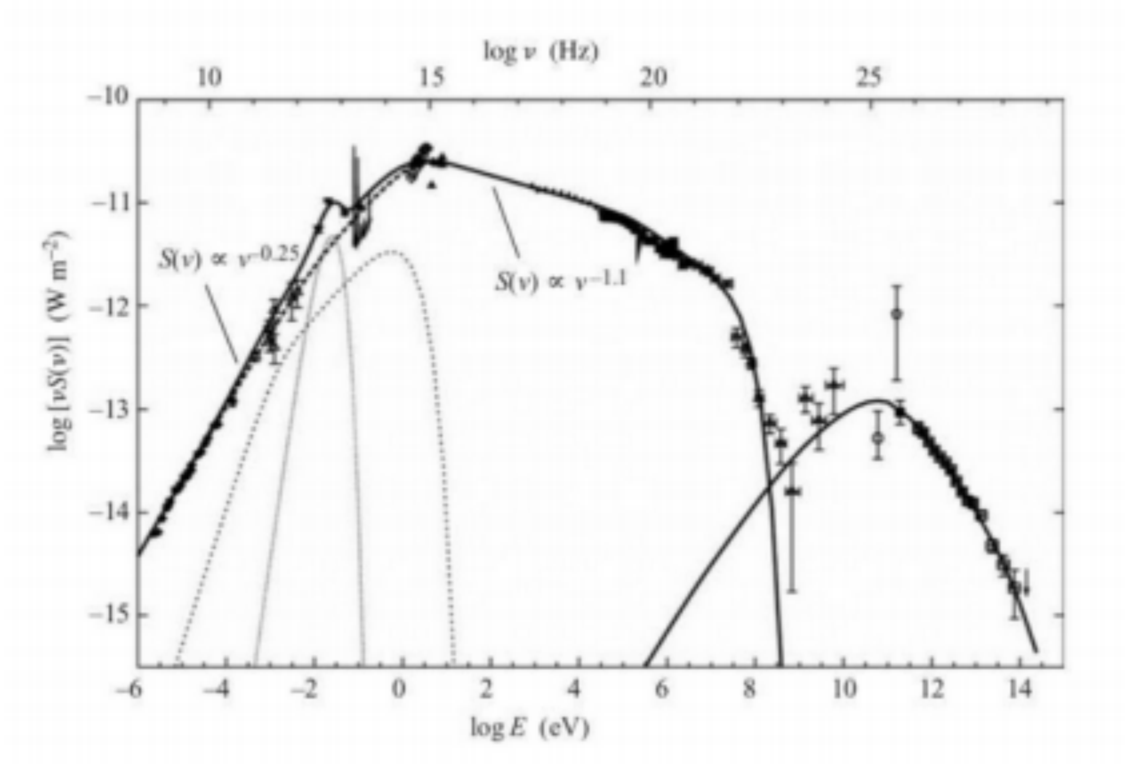
(b) What is the luminosity of the sphere as a function of time after it becomes optically thick?

(c) Give an implicit relation, in terms of R(t), for the time t1 when the sphere becomes optically thick.

(d) Draw a qualitative curve of the luminosity as a function of time.

Problem solving
(synchrotron)

Example 1 : Find an expression for the approximate electron number density ne(cm^{-3}) of synchrotron emitting electrons in radio, optical and x-ray band



Crab Nebula energy spectrum

Problem solving
(synchrotron)

Example 2: Calculate characteristic synchrotron frequency for electrons with Y~10^4 at a magnetic field of 0.1 G. What is the frequency of gyration? Show that in this case photon energy in electrons rest frame is small compared to mc^2. What does this imply?

Problem solving
(synchrotron)

Example 3: What is the typical energy of a scattered photon when 2 GeV cosmic ray electrons interact with the photons of the microwave background radiation, which has a temperature of T = 2.73K. What is the Lorentz factor of incident electrons?

End of Lecture 11

Next Lecture :17th September

Topic of next Lecture:
Compton scattering
(Chapter 7 of Rybicki & Lightman)