Electrodynamics and Radiative Processes I

Lecture 7 – Relativity in Electrodynamics

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Recap Lecture 6

Special theory of relativity

Length contraction (length of a moving rod appears smaller)

Time dilation (moving clock appears slower)

Transformation of velocities

Addition of velocities

Beaming effect

Energy of a moving body

Relativistic Doppler effect

Proper time

Four vectors

Recap Lecture 6

Special theory of relativity

Length contraction (length of a moving rod appears smaller) $L = \left(1 - \frac{v^2}{c^2}\right)^{1/2} L_0$

Time dilation (moving clock appears slower) $T = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma T_0$

Transformation of velocities $u_\parallel = \frac{u'_\parallel + v}{(1 + v u'_\parallel / c^2)}, \quad u_\perp = \frac{u'_\perp}{\gamma(1 + v u'_\parallel / c^2)}$

Addition of velocities

Beaming effect $\theta \sim \frac{1}{\gamma}$

Energy of a moving body $E_k = m_o c^2/\sqrt{(1+v^2/c^2)}$

Relativistic Doppler effect $\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}$

Proper time $c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$

Four vectors

Space-time is a four-vector: $x^\mu = [ct, x]$
For $\mu = 0,1,2,3$

Minkowski

"…Space by it self and time by itself are doomed to fade away into mare shadows, and only a kind of union of the two will preserve an independent reality."

"Goings in the physical world are described by the geometrical structures in the space time"

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

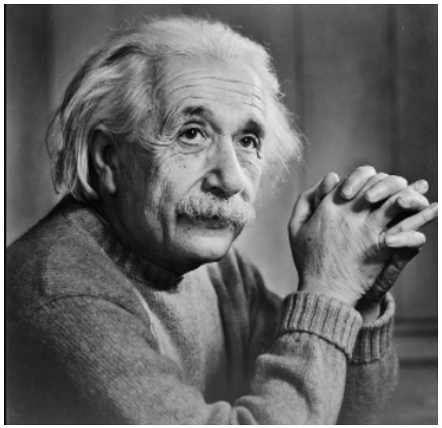
↓

- ✓ Interval between two events is same in all inertial frames of reference. It is invariant under Lorentz transformation.
- ✓ Lorentz transformation is nothing but rotation in space time.

"Minkowski took relativity out of special theory of relativity and presented us with an absolute picture of spatio-temporal activity" @ Penrose

"Minkowski's insights were the key to the discovery of General theory of relativity" @ Penrose

Special theory of relativity to general theory of relativity

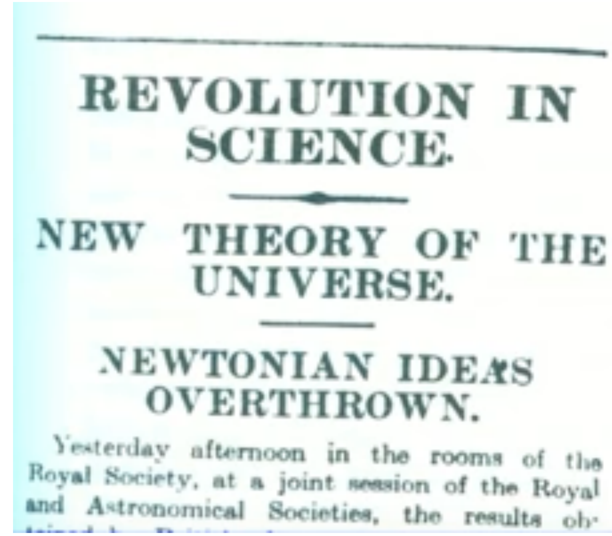


Relation between inertia and energy existed in special theory of relativity
But no relation between inertia and weight..
You can not switch off gravity..

But locally a a freely falling body will bot experience its weight.

Einstein’s thought experiment
“No experiment can distinguish between uniform acceleration due to an engine and uniform to gravitational field.”

Eddington



REVOLUTION IN SCIENCE.

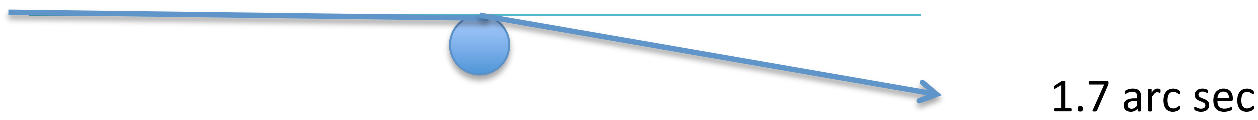
NEW THEORY OF THE UNIVERSE.

NEWTONIAN IDEAS OVERTHROWN.

Yesterday afternoon in the rooms of the Royal Society, at a joint session of the Royal and Astronomical Societies, the results ob-

1919 in a session of the Royal Society of London
Eddington verified Einstein's prediction of diffraction of light by the sun.
Light is deflected according to Einstein's law of Gravitation

"Fusion of two disconnected subjects, metric and gravitation can be considered as the most beautiful achievements of the general theory of relativity." Pauli



Light has weight and would therefore be deflected by gravity

No experiment on Earth will be able to measure this deflection as it is too small and at that time technology would not have allowed it to be measured.

If you have a ray of light grazing the surface of the sun, then it would be deflected by 1.7 arc sec

Gravity modified Minkowski space time..

“Fusion of two disconnected subjects, metric and gravitation can be considered as the most beautiful achievements of the general theory of relativity.” Pauli

Four vectors
(Repeat from Lecture 6)

Four vectors −Four components that transform in a specific way under Lorentz transformation

Length of Four vectors is invariant i.e. same in every inertial system

Electromagnetism predicts that waves travel at c in vacuum.

Laws of electro magnetism must be Lorentz invariant.

Special relativity in one slide
(Repeat from Lecture-6)

Four vector- 1

Space-time is a four-vector: $x^\mu = [ct, \mathbf{x}]$

Four-vectors have Lorentz transformations between two frames with uniform relative velocity v:

$$x' = \gamma(x - \beta ct); \qquad ct' = \gamma(ct - \beta x)$$

$$\beta = v/c \text{ and } \gamma = 1/\sqrt{1-\beta^2}$$

Lengths of four vectors are Lorentz invariant

$$x^\mu x^\nu = c^2t^2 - |x|^2 = c^2t'^2 - |x'|^2 = s^2$$

Charge and Current densities
(Repeat from Lecture-6)

Under a Lorentz transformation a static charge q at rest becomes a charge moving with velocity v. This is a current.

A static charge density ρ at one frame becomes a current density J in other

Note: Charge is conserved by a Lorentz transformation

Four-vector 2
The charge/current four-vector is:

$$J^{\mu} = [c\rho, \mathbf{J}]$$

The full Lorentz transformation is:

$$J'_x = \gamma(J_x - v\rho); \qquad \rho' = \gamma(\rho - v/c^2 J_x)$$

Note: γ factor can be understood as a length contraction or time dilation affecting the charge and current densities

Electrostatic and vector potentials

A static charge density ρ is a source of an electrostatic potential V
A current density J is a source of a magnetic vector potential A

$$V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho}{r} d\tau \qquad \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}}{r} d\tau$$

Four-vector 3
The potential four-vector is

$$A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$$

Under a Lorentz transformation a V becomes an A:

$$A'_x = \gamma(A_x - \frac{v}{c^2}V) \qquad V' = \gamma(V - vA_x)$$

Covarience of electromagnetic phenomenon

Continuity equation:

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \partial_\mu J^\mu = 0$$

This shows that charge conservation is Lorentz invariant!

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Lorentz gauge condition

$$\frac{1}{c}\frac{\partial V}{\partial t} + \nabla . \mathbf{A} = 0 \qquad \partial_\mu A^\mu = 0$$

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$$\frac{1}{c}\frac{\partial V}{\partial t} + \nabla.\mathbf{A} = 0 \qquad \partial_\mu A^\mu = 0$$

Poisson's equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\partial_\mu^2 A^\mu = -\mu_0 J^\mu$$

Electric and Magnetic fields

The Lorentz force on a moving charge is,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

A static point charge is a source of an E field
A moving charge is a current source of a B field

Whether a field is E or B depends on the observer's frame

Lorentz transformation of E and B

Electric and magnetic field in terms of potentials can be written as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz transformation of potentials

$$V' = \gamma(V - vA_x) \qquad A'_x = \gamma(A_x - \frac{v}{c^2}V)$$

Using this transformation and the Lorentz gauge condition the transformations of the electric and magnetic fields are (no derivation)

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \qquad B'_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Lorentz transformation of E and B

A charge at rest has B = 0 and a spherically symmetric E field
A highly relativistic charge has $\beta \rightarrow 1$, $\gamma >> 1$

The electric field is

$$E'_x = E_x \ll |\mathbf{E}'| \qquad E'_y = \gamma E_y \qquad E'_z = \gamma E_z$$

The magnetic field is

$$B'_x = 0 \qquad B'_y = \gamma \frac{v}{c^2} E_z \qquad B'_z = -\gamma \frac{v}{c^2} E_y$$

Electromagnetic field Tensor

Electric field and magnetic field can be expressed as components of Electromagnetic field tensor in following form

$$F^{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$$

where $x = [ct, \mathbf{x}]$ $\quad A = [V/c, \mathbf{A}]$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell's equation in terms of $F^{\mu\nu}$

Maxwell's equation with **source terms**

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = J^\mu$$

M1 $\nabla.\mathbf{E} = \rho/\epsilon_0$ $\mu = 0,\ \nu = (1,2,3)$

M4 $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \partial\mathbf{E}/\partial t)$ $\mu = 1,\ \nu = (2,3,0)$

Maxwell's equation without **source terms**

$$\frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu} = 0$$

M2 $\nabla.\mathbf{B} = 0$ $(\mu,\nu,\sigma) = (1,2,3)$

M3 $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ $(\mu,\nu,\sigma) = (0,1,2)(3,0,1)(2,3,0)$

Maxwell's equations are Lorentz invariant

Relativity and electromagnetic field

For the pure boost with velocity **v**=c**β**, equations can be written in the following form

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta}\times\mathbf{B}) \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta}\times\mathbf{E})$$

Pure Electric field is not Lorentz invariant

Pure Magnetic field is not Lorentz invariant

e.g. If the field is purely electric in one frame in another frame it will be a mixed electric and magnetic field

Any scalar formed with $F^{\mu\nu}$ represents function of E and B that is Lorentz invariant.

Relativity and electromagnetic field

Any scalar formed with $F^{\mu\nu}$ represents function of E and B that is Lorentz invariant.

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2). \longrightarrow \mathbf{B}^2 - \mathbf{E}^2 = \mathbf{B}'^2 - \mathbf{E}'^2$$

$$\det F = (\mathbf{E}\cdot\mathbf{B})^2. \longrightarrow \mathbf{E}\cdot\mathbf{B} = \mathbf{E}'\cdot\mathbf{B}'$$

Fields of a uniformly moving charge

Fields of a charge moving with constant velocity v in the x-axis

In the rest frame of the charge the fields are

$$E'_x = \frac{qx'}{r'^3} \qquad B'_x = 0$$

$$E'_y = \frac{qy'}{r'^3} \qquad B'_y = 0$$

$$E'_z = \frac{qz'}{r'^3} \qquad B'_z = 0$$

$$r'^3 = \left(x'^2 + y'^2 + z'^2\right)^{3/2}.$$

Fields in the moving frame of charge

$$E_x = \frac{qx'}{r'^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y'}{r'^3} \qquad B_y = -\frac{q\gamma\beta z'}{r'^3}$$

$$E_z = \frac{q\gamma z'}{r'^3} \qquad B_z = \frac{q\gamma\beta y'}{r'^3}.$$

$\Rightarrow$

$$E_x = \frac{q\gamma(x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3}$$

$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

$$r^3 = \left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{3/2}.$$

Fields of a uniformly moving charge

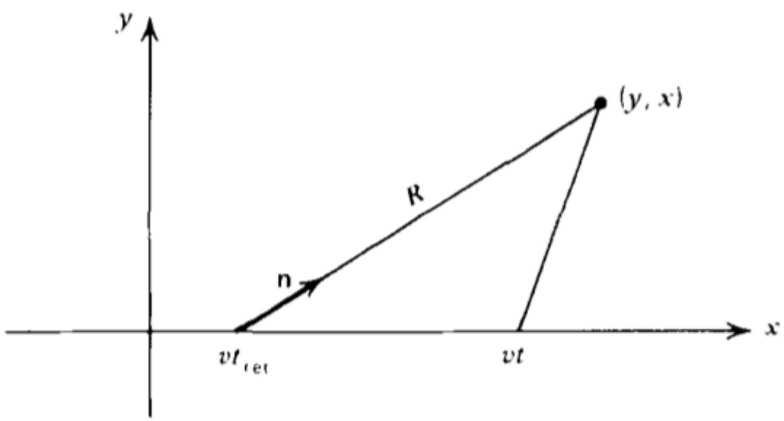
$$E_x = \frac{q\gamma(x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3}$$

$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

$$\mathbf{E} = q\left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2}\right]$$

E derived from
Lienard-Wiechert potential
in near field regime
(refer to Rybicki and Lightman
for derivation)



$$t_{ret} = t - \frac{R}{c}$$

Fields of a uniformly moving charge

Consider the case of highly relativistic charge γ>>1

Consider the field point at a distance b from the origin along y axis (x=0,z=0).

$$E_x = \frac{q\gamma(x - vt)}{r^3} \qquad B_x = 0$$

$$E_y = \frac{q\gamma y}{r^3} \qquad B_y = -\frac{q\gamma\beta z}{r^3}$$

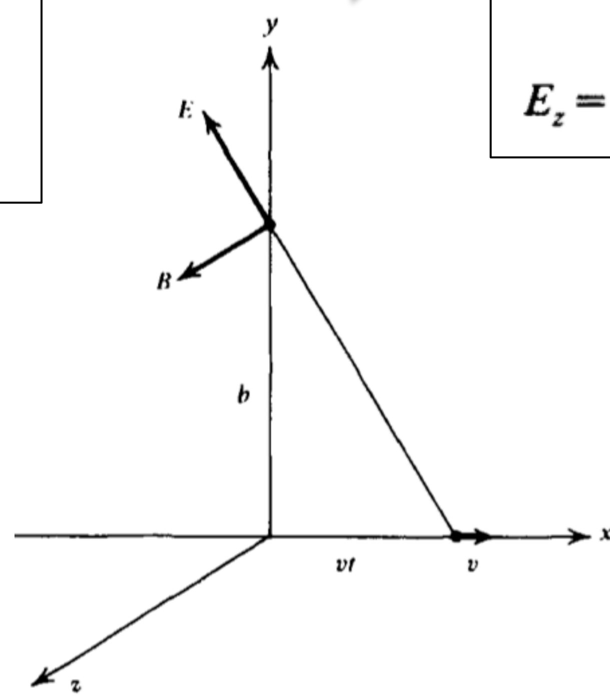
$$E_z = \frac{q\gamma z}{r^3} \qquad B_z = \frac{q\gamma\beta y}{r^3}$$

→

$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_x = 0$$

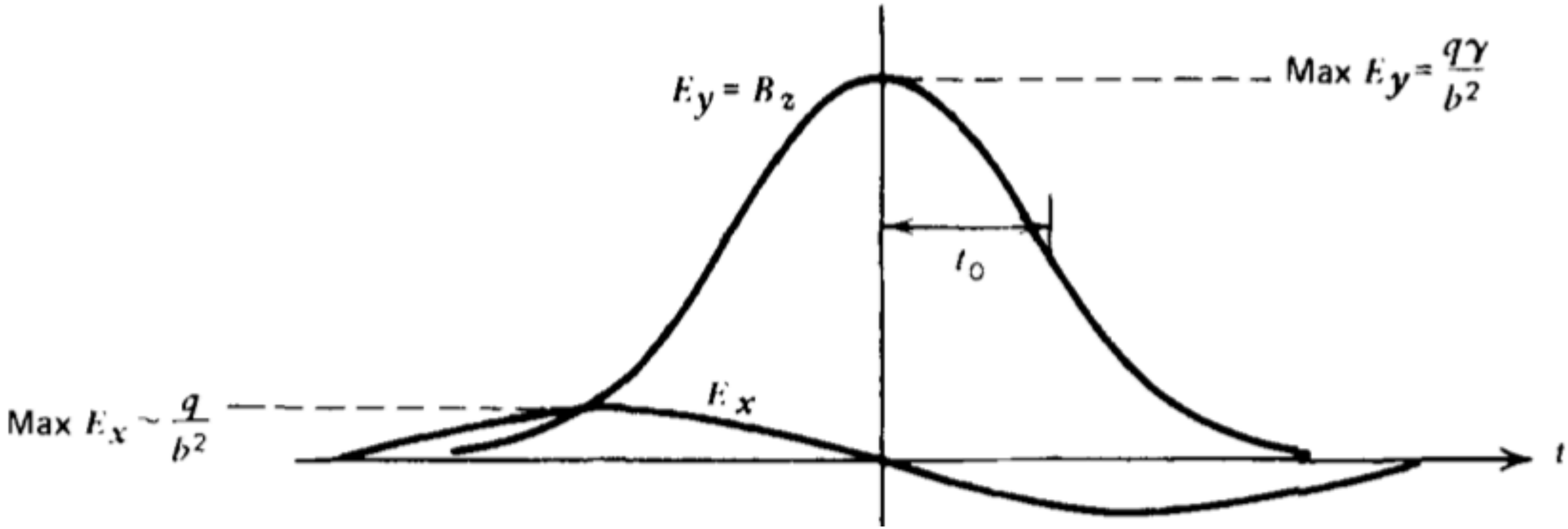
$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_y = 0$$

$$E_z = 0 \qquad B_z = \beta E_y$$



Electric and magnetic fields from a uniformly moving particle

Fields of a uniformly moving charge



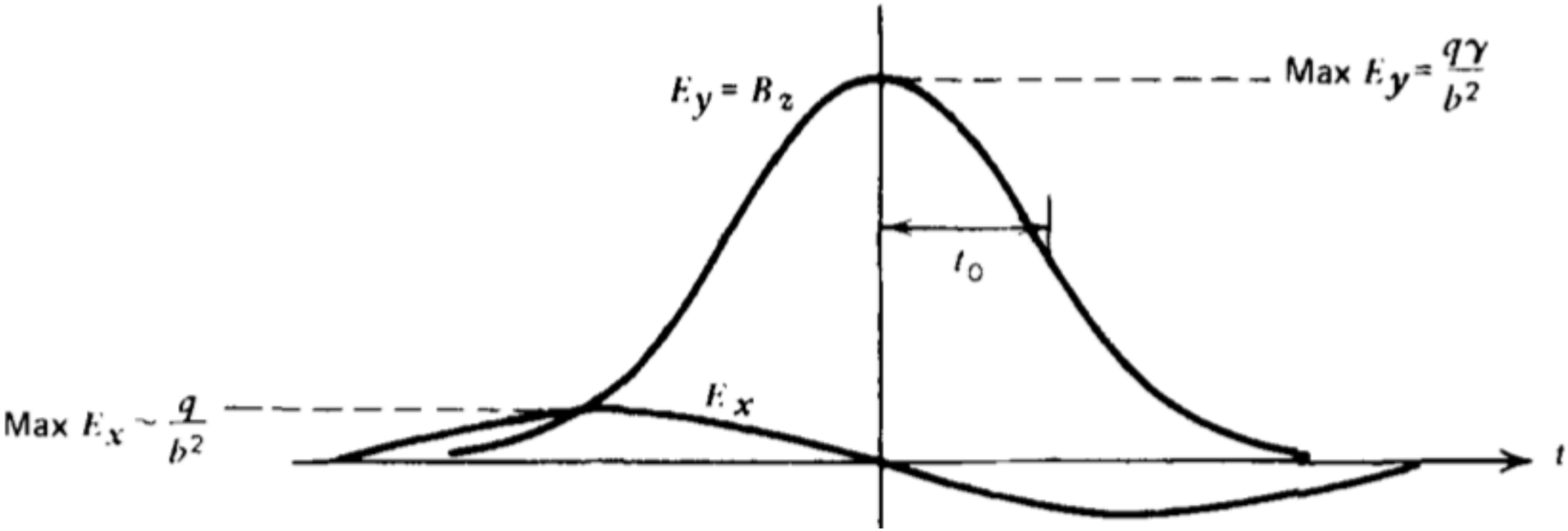
Time dependence of fields E_x and E_y from a particle of uniform high velocity

$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_x = 0$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \qquad B_y = 0$$

$$E_z = 0 \qquad B_z = \beta E_y$$

Fields of a uniformly moving charge

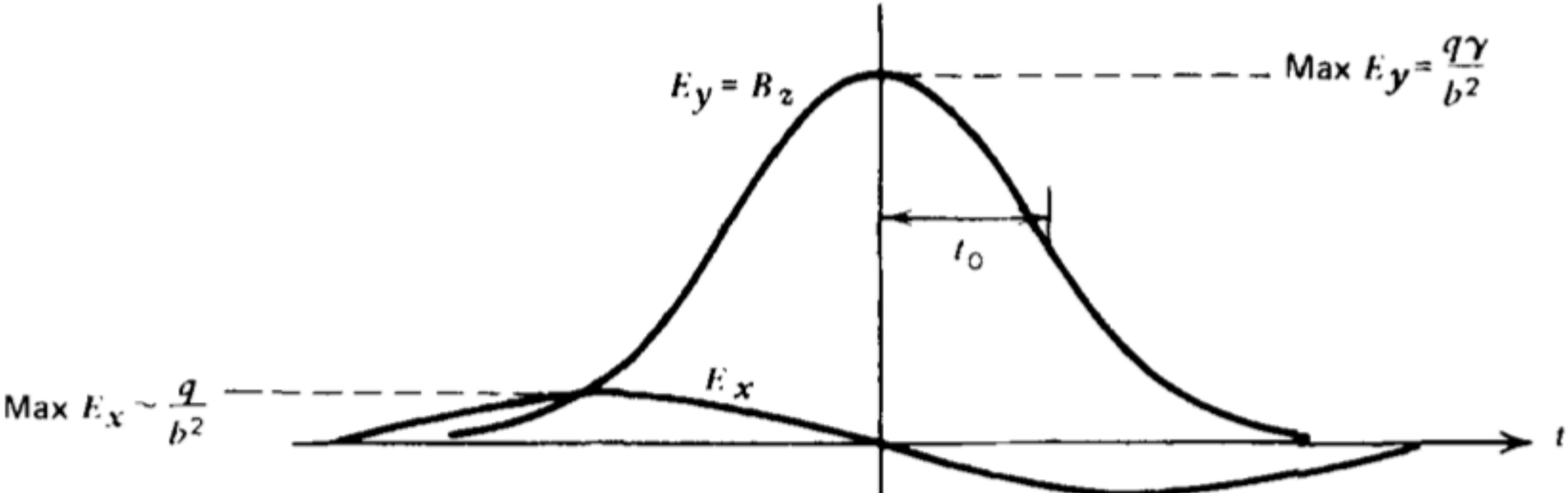


Time dependence of fields E_x and E_y from a particle of uniform high velocity

✓ Fields are strong when t~b/γv

✓ Fields of the moving charges are concentrated in the plane transverse to its motion into an angle of order of 1/γ

Fields of a uniformly moving charge



Time dependence of fields E_x and E_y from a particle of uniform high velocity

✓ Fields are mostly transverse (in y direction) since (Max E_x)/(Max E_y)= Y

✓ Field of a highly relativistic charge will appear as a pulse of radiation traveling in The same direction as the charge and confined to the transverse plane: Connection between fields of used in treatment of “method of virtual quanta” in relativistic bremssrahlung /synchrotron radiation.

Equivalent spectra

$$\hat{E}(\omega) = \frac{1}{2\pi}\int E_2(t)e^{i\omega t}\,dt$$

$$= \frac{q\gamma b}{2\pi}\int_{-\infty}^{\infty}(\gamma^2 v^2 t^2 + b^2)^{-3/2} e^{i\omega t}\,dt$$

Integration in terms of modified Bessel function,

$$\hat{E}(\omega) = \frac{q}{\pi b v}\frac{b\omega}{\gamma v}K_1\left(\frac{b\omega}{\gamma v}\right)$$

Thus the spectrum is,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$

Equivalent spectra

The spectrum of virtual pulse,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$

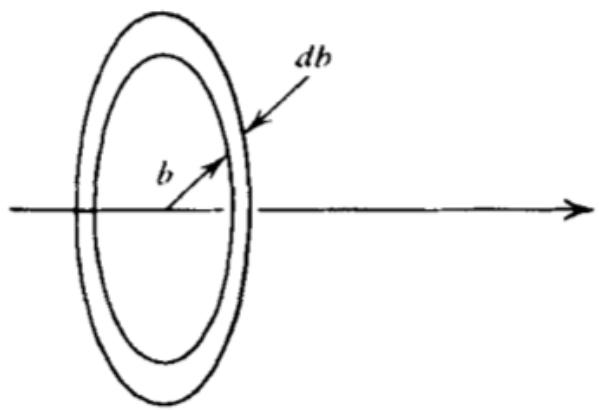
The spectrum starts to cut off for $\omega > \gamma v/b$

(From uncertainty principle since pulse is confined to time interval of $\sim b/\gamma v$)

Equivalent spectra

The spectrum of virtual pulse,

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = \frac{q^2 c}{\pi^2 b^2 v^2}\left(\frac{b\omega}{\gamma v}\right)^2 K_1^2\left(\frac{b\omega}{\gamma v}\right)$$



Area element perpendicular to the velocity of a moving particle $dA = 2\pi b\,db$

Energy per unit frequency range

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA\,d\omega}\, b\,db$$

Equivalent spectra

Energy per unit frequency range

$$\frac{dW}{d\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{dW}{dA\, d\omega}\, b\, db$$

Lower limit to satisfy the description of field in classical electrodynamics and considering point charge,

e.g. b_{min}=radius of ion

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \int_x^{\infty} y K_1^2(y)\, dy$$

Considering

$$y \equiv \frac{\omega b}{\gamma v} \qquad x \equiv \frac{\omega b_{\min}}{\gamma v}$$

Integrating in terms of Bessel functions

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2} \left[x K_0(x) K_1(x) - \tfrac{1}{2} x^2 \left(K_1^2(x) - K_0^2(x) \right) \right]$$

Equivalent spectra

Energy per unit frequency range

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\left[xK_0(x)K_1(x) - \tfrac{1}{2}x^2\left(K_1^2(x) - K_0^2(x)\right)\right]$$

Two limits when ω is small and large

$$\frac{dW}{d\omega} = \frac{q^2c}{2v^2}\exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right), \quad \omega \gg \frac{\gamma v}{b_{\min}}$$

$$\frac{dW}{d\omega} = \frac{2q^2c}{\pi v^2}\ln\left(\frac{0.68\gamma v}{\omega b_{\min}}\right), \qquad \omega \ll \frac{\gamma v}{b_{\min}}$$

Emission from relativistic particles

The energy in as frame K moving with velocity −v with respect to the particle is

$$dW = \gamma\, dW', \qquad dt = \gamma\, dt';$$

Total power emitted in frames K and K'

$$P = \frac{dW}{dt}, \qquad P' = \frac{dW'}{dt'} \quad \Rightarrow \quad P = P'.$$

Emitted power is Lorentz invariant for any emitter that emits with Front-back symmetry in its instantaneous rest frame

Emission from relativistic particles

$$P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2$$

$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}.$$

The power to be calculated in any frame by calculating **a** in that particular frame and squaring

$$a'_{\parallel} = \gamma^3 a_{\parallel},$$

$$a'_{\perp} = \gamma^2 a_{\perp}.$$

(Rybicki & Lightman 4.3)

Emission from relativistic particles

$$P' = \frac{2q^2}{3c^3}|\mathbf{a}'|^2$$

$$P = \frac{2q^2}{3c^3}\vec{a}\cdot\vec{a}.$$

The power and be calculated in any frame by calculating a in that particular frame and squaring

$$P = \frac{2q^2}{3c^3}\mathbf{a}'\cdot\mathbf{a}' = \frac{2q^2}{3c^3}\left(a_{\parallel}'^2 + a_{\perp}'^2\right)$$
$$= \frac{2q^2}{3c^3}\gamma^4\left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right)$$

$$a_{\parallel}' = \gamma^3 a_{\parallel},$$
$$a_{\perp}' = \gamma^2 a_{\perp}.$$

End of Lecture 7

"It's not that I'm so smart, it's just that I stay with problems longer."
@ Einstein

14th September (Friday) 2:30-3:30 is OK for lecture?

Next Lecture : 4th August
(no Lecture on 3rd as it is Holiday)