Electrodynamics and Radiative Processes I

Lecture 6 – Radiation from moving charges

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August-September 2018

Date : 23th August 2018

Recap Lecture 5

Maxwell's equation with source terms

Introduce scalar potential Φ(r,t) and vector potential **A(r,t)**

Expression of Φ(r,t) and A(r,t) in terms of κ and R at retarded time

Expression of Electric field E having two components

Velocity field and Radiation field and when they are important

Total power radiated by non relativistic point charge when it accelerates

Dipole approximation

Recap Lecture 5

Maxwell's equation with source terms

Introduce scalar potential Φ(r,t) and vector potential **A(r,t)**

Expression of Φ(r,t) and A(r,t) in terms of κ and R at retarded time

$$\phi = \left[\frac{q}{\kappa R} \right]$$

$$\mathbf{A} = \left[\frac{q\mathbf{u}}{c\kappa R} \right]$$

Expression of Electric field E having two components

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2} \right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\} \right]$$

Velocity field and Radiation field and when they are important

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{Ruv}{c^2} = \frac{u}{c}\frac{R}{\lambda}.$$

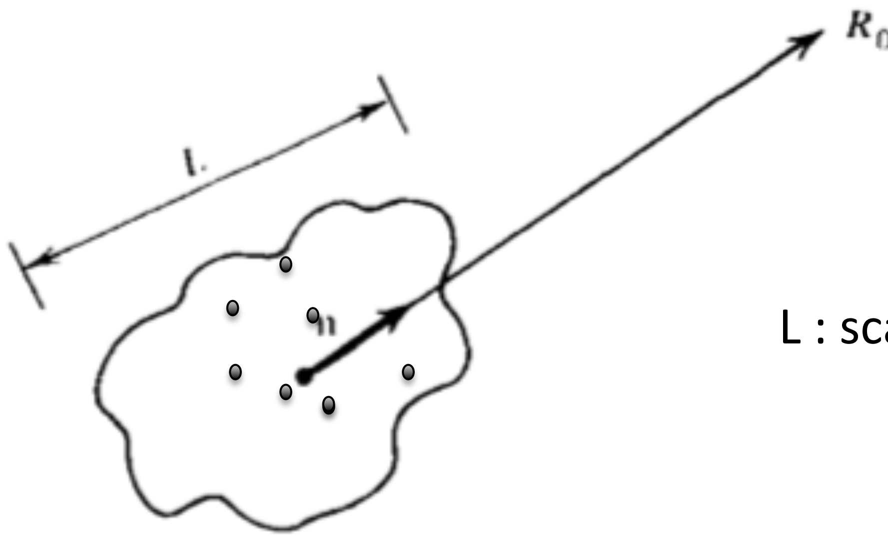
Total power radiated by non relativistic point charge when it accelerates

$$P = \frac{2q^2\dot{u}^2}{3c^3}$$

Dipole approximation

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Dipole approximation



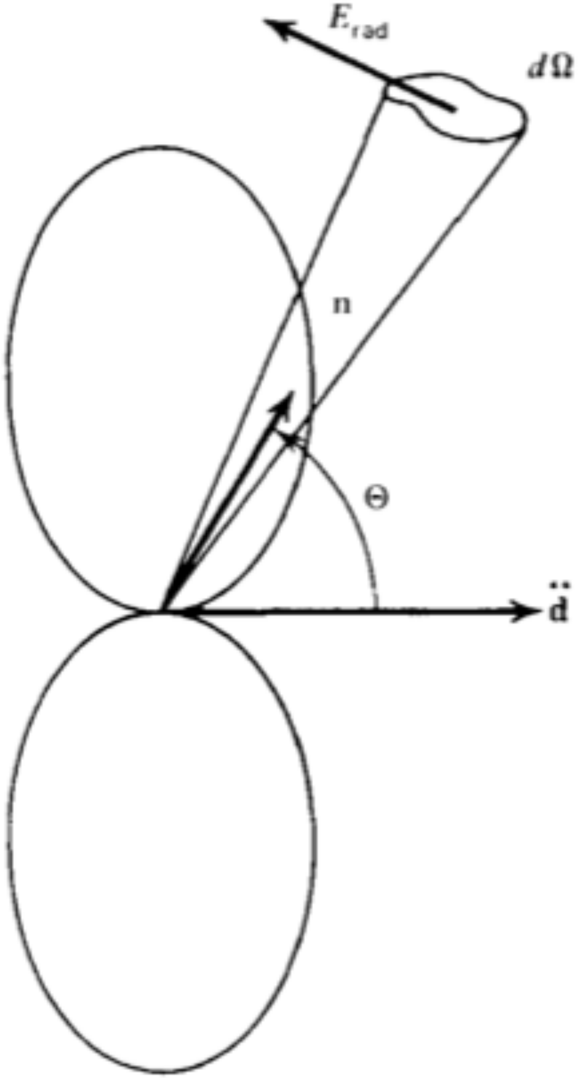
L : scale of the system

Differences in retarded time can be ignored if
size of the system is small compared to wavelength

⟺ $\lambda \gg L$

Dipole approximation

dp/dΩ



$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta,$$

- ✓ Pattern is symmetric about dipole moment/acceleration
- ✓ Independent of velocity
- ✓ Intensity of radiation is zero along the direction of acceleration.

Angular distribution of dipole radiation

Radiation pattern of a half wave dipole antenna

Dipole approximation

$$\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}_i)}{R_i}$$

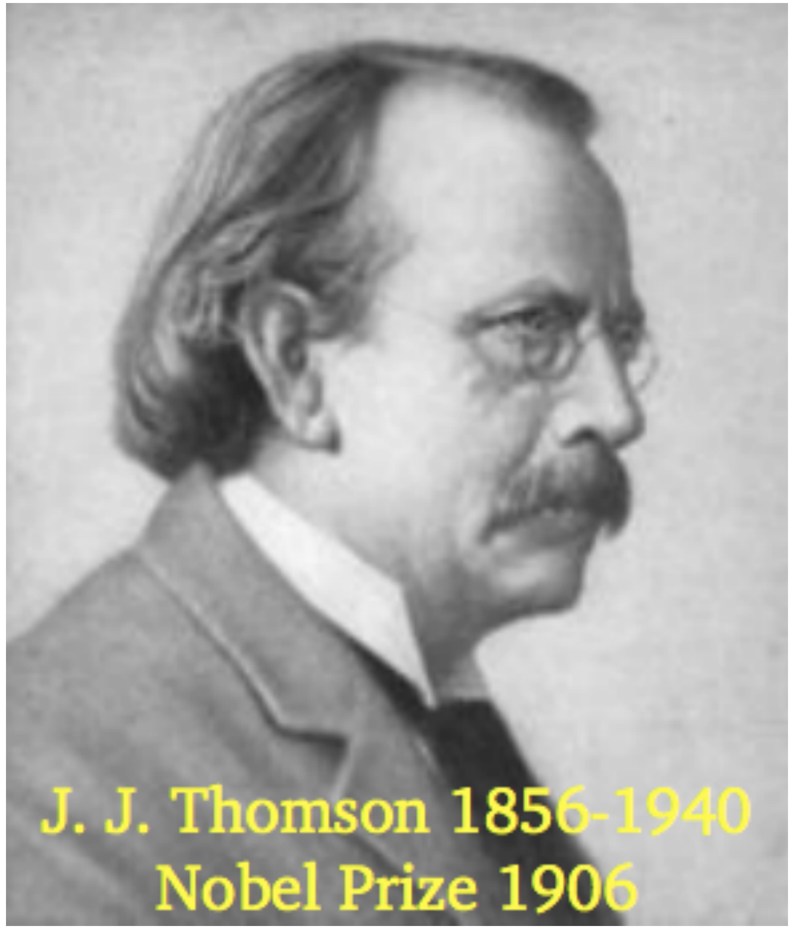
$$\mathbf{d} = \sum_i q_i \mathbf{r}_i.$$

$$\mathbf{E}_{\text{rad}} = \frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{d}})}{c^2 R_0}$$

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3}\sin^2\Theta,$$

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

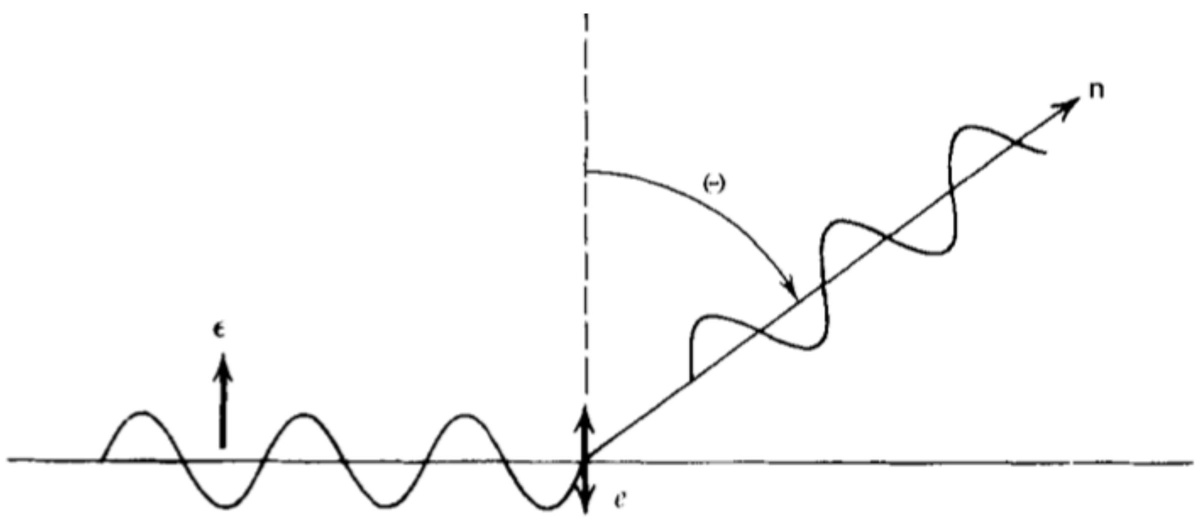
Dipole approximation :
Larmor's formula extended for a collection of non-relativistic particles



"In recognition of the great merits of his theoretical and experimental investigations on the conduction of electricity by gases."

Thomson scattering

Let us consider application of the dipole formula in a process in which a free charge radiates in response to an incident electromagnetic wave



Process by which an electromagnetic wave is scattered in to random directions by a free electron.

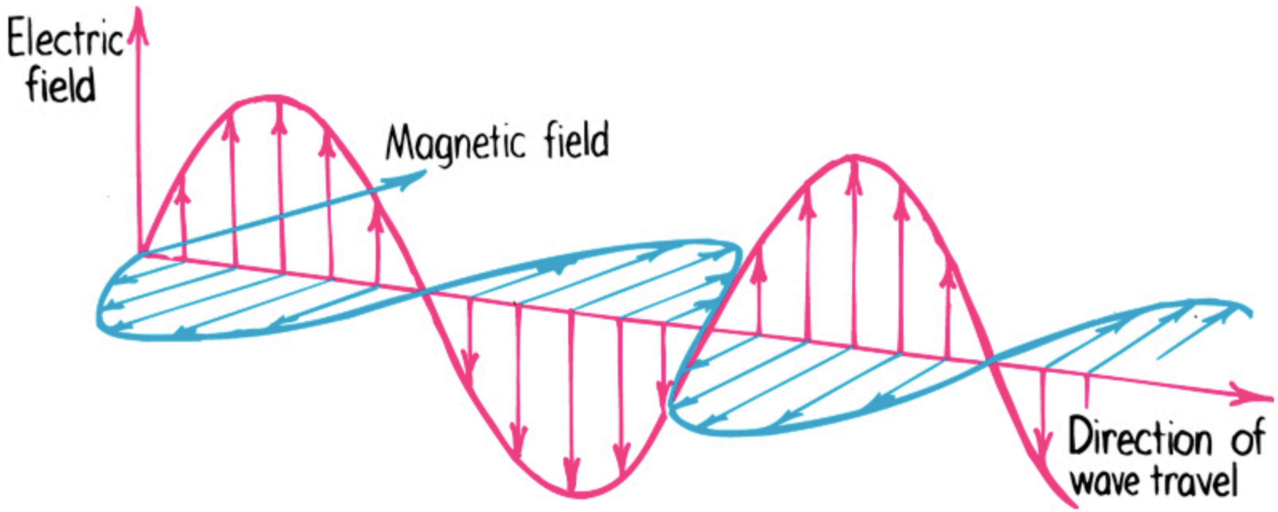
Applicable for $h\nu \ll m_e c^2$

Thomson scattering

Process by which an electromagnetic wave is scattered in to random directions by a free electron.

Applicable for $h\nu << m_e c^2$

Consider a linearly polarized electromagnetic wave incident on a free electron



Force on the electron

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

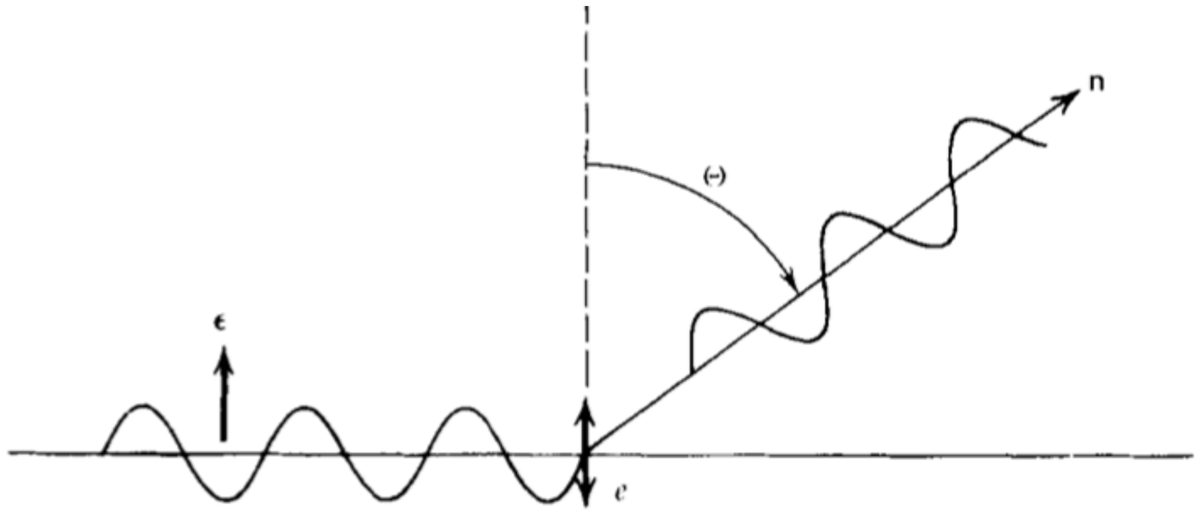
Negligible as v<<c

Thomson scattering

Force of a linearly polarized wave acting on a electron

$$\mathbf{F} = e\boldsymbol{\epsilon} E_0 \sin \omega_0 t$$

$$m\ddot{\mathbf{r}} = e\boldsymbol{\epsilon} E_0 \sin \omega_0 t$$

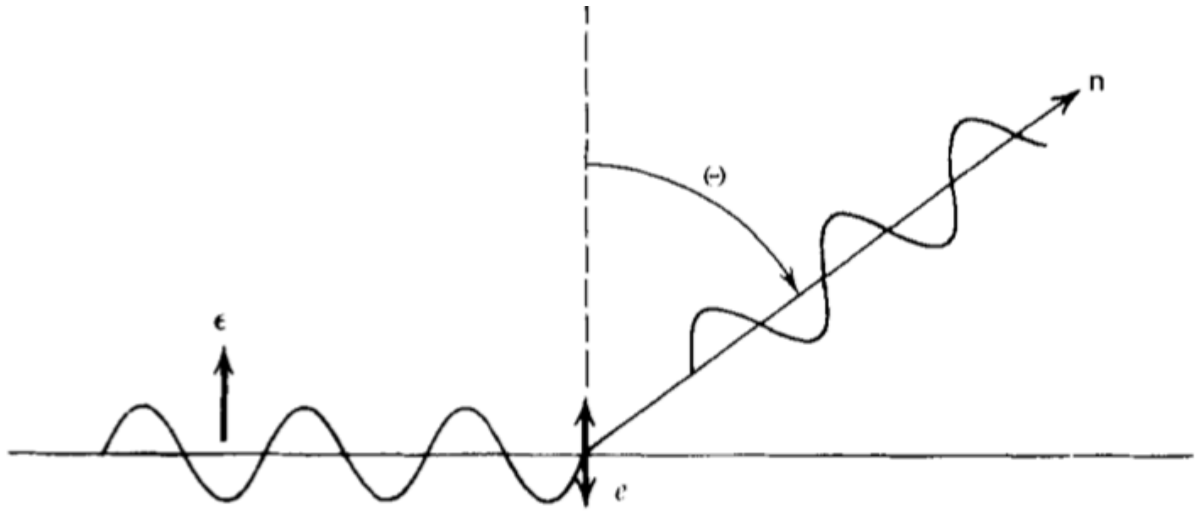


Thomson scattering

Force of a linearly polarized wave acting on a electron

$$\mathbf{F} = e\boldsymbol{\epsilon} E_0 \sin \omega_0 t.$$

$$m\ddot{\mathbf{r}} = e\boldsymbol{\epsilon} E_0 \sin \omega_0 t.$$



Dipole moment is defined by $\mathbf{d} = e\mathbf{r},$

$$\ddot{\mathbf{d}} = \frac{e^2 E_0}{m} \boldsymbol{\epsilon} \sin \omega_0 t,$$

$$\mathbf{d} = -\left(\frac{e^2 E_0}{m\omega_0^2}\right) \boldsymbol{\epsilon} \sin \omega_0 t,$$

Oscillating dipole of amplitude $\mathbf{d}_0 = \frac{e^2 E_0}{m\omega_0^2} \boldsymbol{\epsilon}.$

Thomson scattering

Dipole approximation

Radiation from a non relativistic system of particles (λ>>L)

Second derivative of dipole moment

$$\mathbf{d} = \sum_i q_i \mathbf{r}_i.$$

Power radiated per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2\Theta,$$

Total Power radiated

$$P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}$$

Thomson scattering

Electron subject to electromagnetic wave (hν<<mc²)

Second derivative of dipole moment

$$\ddot{\mathbf{d}} = \frac{e^2 E_0}{m} \boldsymbol{\epsilon} \sin \omega_0 t,$$

Power radiated per unit solid angle

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2\Theta$$

(time average of $\sin^2 \omega_o t$ gives a factor ½)

Total Power radiated

$$P = \frac{e^4 E_0^2}{3m^2 c^3}$$

Thomson scattering

Electron subject to electromagnetic wave

Remember time averaged pointing flux is defined as $\langle S \rangle = \frac{c}{8\pi} E_0^2$

Define differential cross section $d\sigma$ for scattering in to $d\Omega$

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi} \frac{d\sigma}{d\Omega}$$

Thomson scattering

Define differential cross section $d\sigma$ for scattering in to $d\Omega$

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi} \frac{d\sigma}{d\Omega}$$

$$\frac{dP}{d\Omega} = \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2\Theta$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{polarized}} = \frac{e^4}{m^2 c^4} \sin^2\Theta = r_0^2 \sin^2\Theta$$

$$r_0 \equiv \frac{e^2}{mc^2}$$

→ **Classical electron radius**

Thomson scattering

Classical electron radius

$$r_0 \equiv \frac{e^2}{mc^2}$$

Measure of the size of the point charge
(assuming the rest energy is purely electromagnetic)
$\sim 2.8 \times 10^{-13}$ cm

Total cross-section is obtained after integrating over solid angle,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1-\mu^2) d\mu = \frac{8\pi}{3} r_0^2 = \sigma_T \sim 0.66 \times 10^{-24} \text{ cm}^2$$

↓

Thomson Scattering cross section

Frequency independent, so scattering is equally effective at all frequencies.
Valid for lower frequencies
Not valid for high frequencies $h\nu > mc^2$

Thomson scattering

Total cross-section is obtained after integrating over solid angle,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi r_0^2 \int_{-1}^{1} (1-\mu^2) d\mu = \frac{8\pi}{3} r_0^2 = \sigma_T \sim 6.65 \times 10^{-25}\ \text{cm}^2$$

↓

Thomson Scattering cross section

Frequency independent, so scattering is equally effective at all frequencies.

Valid for lower frequencies where $h\nu \ll mc^2$

Not valid for high frequencies when $h\nu$ is comparable or larger than mc^2

For very intense radiation fields electron moves with relativistic velocity and dipole approximation is not valid

Thomson scattering

Calculated Thomson scattering cross-section for an electron and Polarized EM wave

↓

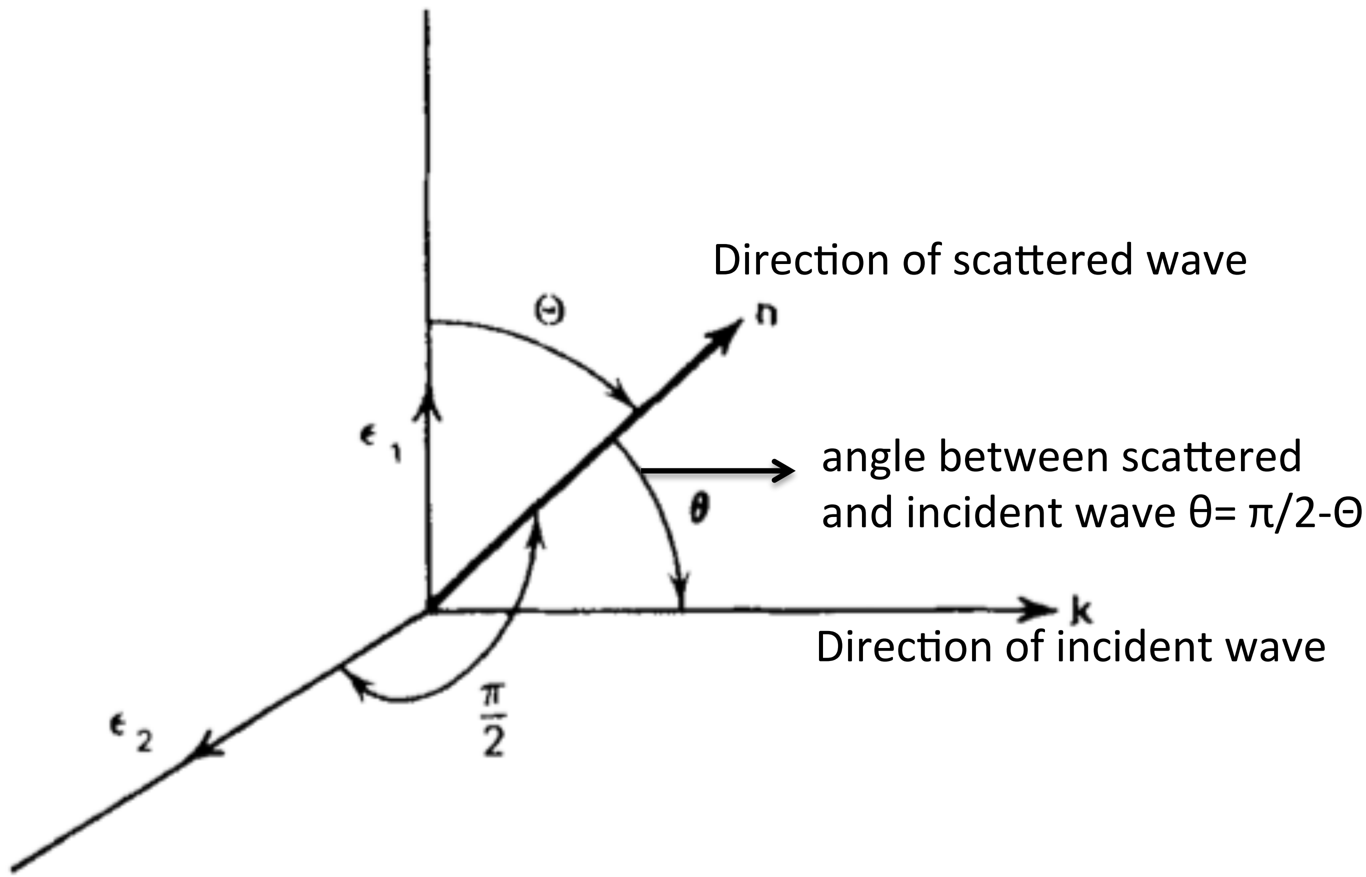
Incoming wave linearly polarized along **ε**

Outgoing EM wave is also linearly polarized in the plane defined by **ε** and **n**

Unpolarized EM (better randomly polarized) wave can be regarded as superposition of two linearly polarized beams with perpendicular axes

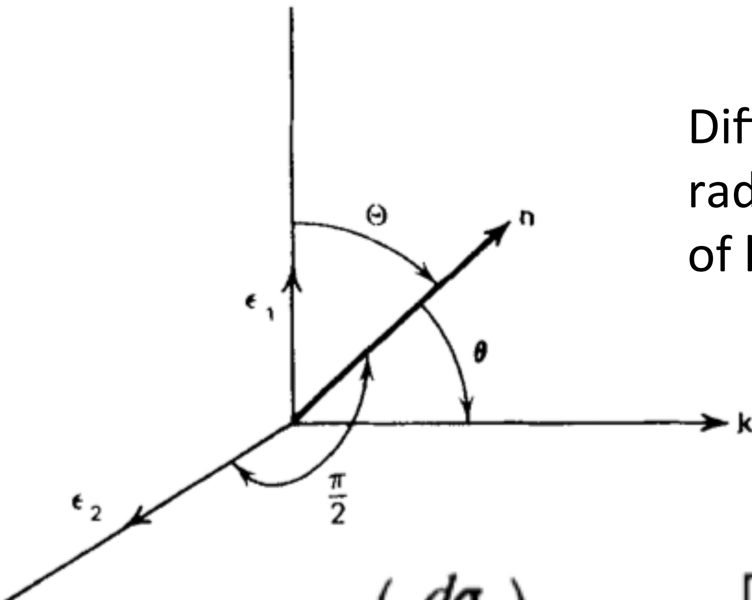
Thomson scattering

Unpolarized EM wave can be regarded as superposition of two linearly polarized beams with perpendicular axes ε_1 and ε_2



Thomson scattering

Unpolarized EM wave can be regarded as superposition of two linearly polarized beams with perpendicular axes



Differential cross section for unpolarized radiation is the average of the cross sections of linear-polarized radiation through Θ and π/2

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right]$$

$$= \frac{1}{2} r_0^2 (1 + \sin^2\Theta)$$

$$= \frac{1}{2} r_0^2 (1 + \cos^2\theta)$$

Thomson scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right]$$

✓ Forward-backward symmetry : The scattering cross section is symmetric under the reflection θ → -θ

✓ Total cross section: The total scattering cross-section of unpolarized incident radiation is same as that for polarized incident radiation.
Since electron at rest has no direction intrinsically defined.

$$\sigma_{\text{unpol}} = \sigma_{\text{pol}} = (8\pi/3)r_0^2$$

Thomson scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right]$$

intensities in two perpendicular directions in the plane normal to **n** arising from two perpendicular components of the incident wave

Polarization intensities in the plane and perpendicular are $\cos^2\theta : 1$

For partially polarized light degree of polarization of the scattered wave

$$\Pi = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \longrightarrow \Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta}$$

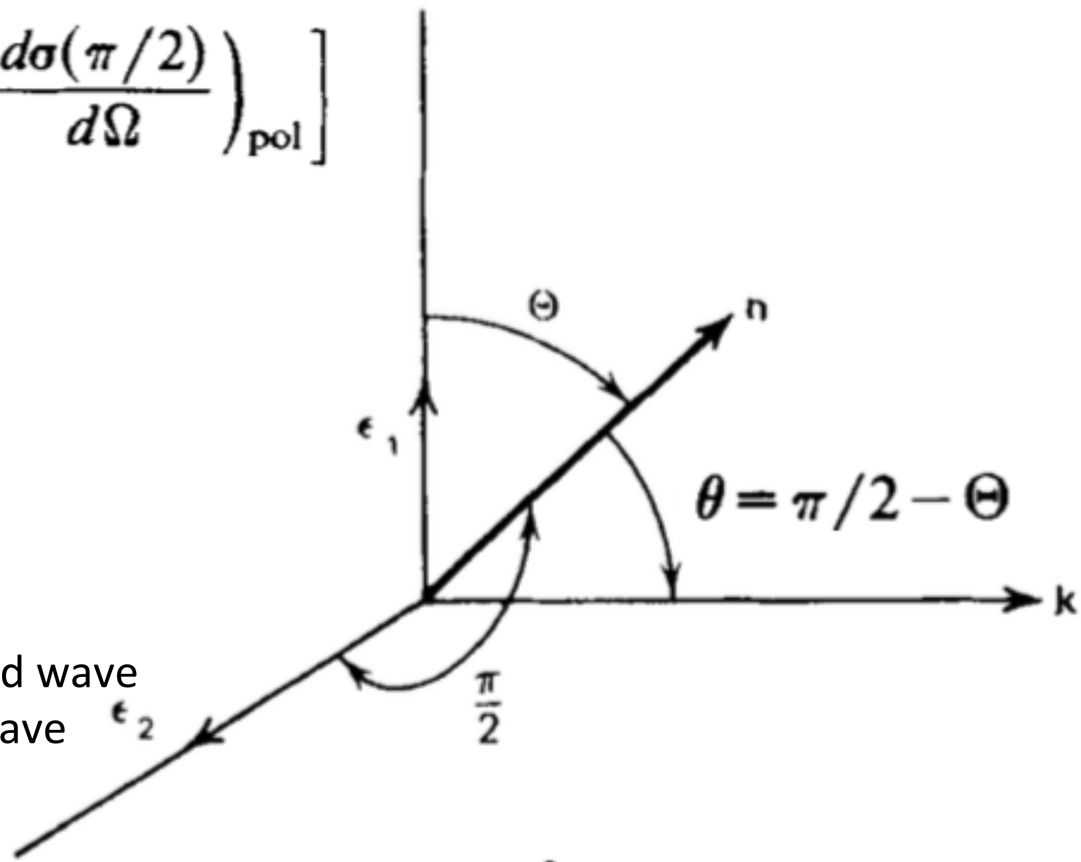
Thomson scattering

k direction of incoming e.m. wave

n direction of scattered wave

Total scattering cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right]$$
$$= \frac{1}{2} r_0^2 (1+\sin^2\Theta)$$
$$= \frac{1}{2} r_0^2 (1+\cos^2\theta),$$



Reflection θ → -θ

Scattering cross-section is same

✓ Scattering cross section for unpolarized wave
= Scattering cross-section for polarized wave

$$\sigma_{\text{unpol}} = \sigma_{\text{pol}} = (8\pi/3) r_0^2.$$

✓ Degree of polarization of scattered wave

$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta}$$

Thomson scattering

$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta}$$

Since $\Pi > 0$ electron scattering of a completely unpolarized incident wave produces scattered wave with some degree of polarization intensities. The degree depend on θ

Example

Absorption coefficient

$$\alpha = n\sigma_T$$

Number density of particles

Thomson scattering cross section $\sim 6.62 \times 10^{-25}\ \mathrm{cm}^2$

So Thompson scattering is significant only when number density is high

The cosmic microwave background is linearly polarized as a result of Thomson scattering (as measured by Degree angular scale interferometer(DASI) and more recent experiments).

The solar K-corona is the result of the Thomson scattering of solar radiation from solar coronal electrons.

Example

Optical depth

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\, ds'.$$

$$\tau = n \sigma_T R$$

Now considering a nebula having n =10,000 and
At a distance of $R=10^{19}$ cm

Then we can get estimate of $\tau = 10{,}000 \times 10^{19} \times 6.25 \times 10^{-25} = 0.07$

Optically thin

Example

The cross-section for Thomson scattering is tiny and therefore Thomson scattering is most important when the density of free electrons is high, as in the early Universe or in the dense interiors of stars.

Radiation reaction

Force acting on a particle by virtue of the radiation it produces

→ Radiation reaction force

Let T be the time interval over which kinetic energy of the particle is changed substantially by the emission of radiation

$$T \sim \frac{mv^2}{P_{rad}} \sim \frac{3mc^3}{2e^2}\left(\frac{v}{a}\right)^2$$

$$P = \frac{2q^2\dot{u}^2}{3c^3}$$

$1/\tau$

$$\tau \equiv \frac{2e^2}{3mc^3} \sim 10^{-23} s$$

Radiation reaction

$$\tau \equiv \frac{2e^2}{3mc^3} \sim 10^{-23} s$$

$\tau \sim r_o/c$ → time for radiation to cross a distance comparable to classical electron radius

As long as we are considering processes that occur on a time scale much longer than τ, we can treat radiation reaction as a perturbation.

Radiation reaction

Energy radiated compensated by work done against radiation reaction force F_{rad}.

$$-\mathbf{F}_{rad}\cdot\mathbf{u} = \frac{2e^2\dot{u}^2}{3c^3}$$

$$-\int_{t_1}^{t_2}\left(\mathbf{F}_{rad} - \frac{2e^2\ddot{\mathbf{u}}}{3c^3}\right)\cdot\mathbf{u}\,dt = 0.$$

$$\mathbf{F}_{rad} = \frac{2e^2\ddot{\mathbf{u}}}{3c^3} = m\tau\ddot{\mathbf{u}},$$

↓

Abraham–Lorentz force

Radiation reaction

$$\mathbf{F}_{\text{rad}} = \frac{2e^2\ddot{\mathbf{u}}}{3c^3} = m\tau\ddot{\mathbf{u}},$$

Recoil force acting on the charge

Proportional to the acceleration

Valid for non relativistic cases.
Dirac proposed relativistic version

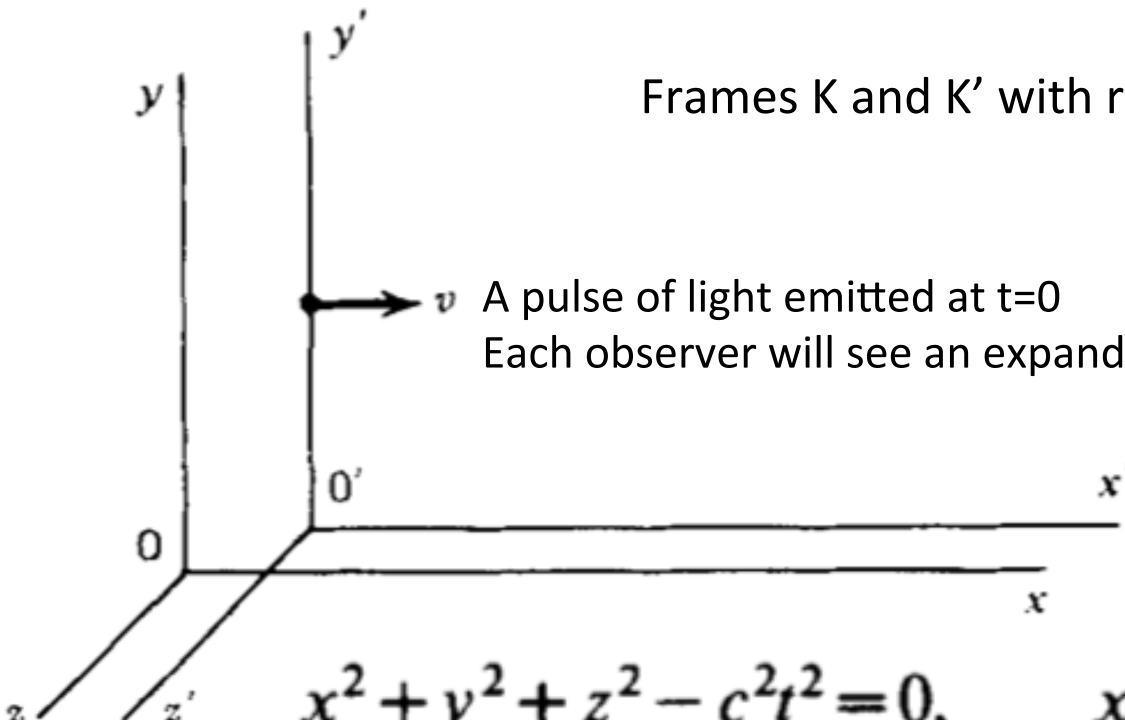
An accelerating charge emits radiation according to the Larmor formula, which carries momentum away from the charge.

But momentum is conserved, so the charge is pushed in the direction opposite the direction of the emitted radiation: radiation reaction.

Review of Lorentz transformation and four vectors

Review of Lorentz Transformations

✓ The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.
✓ The speed of light is c in all such frames



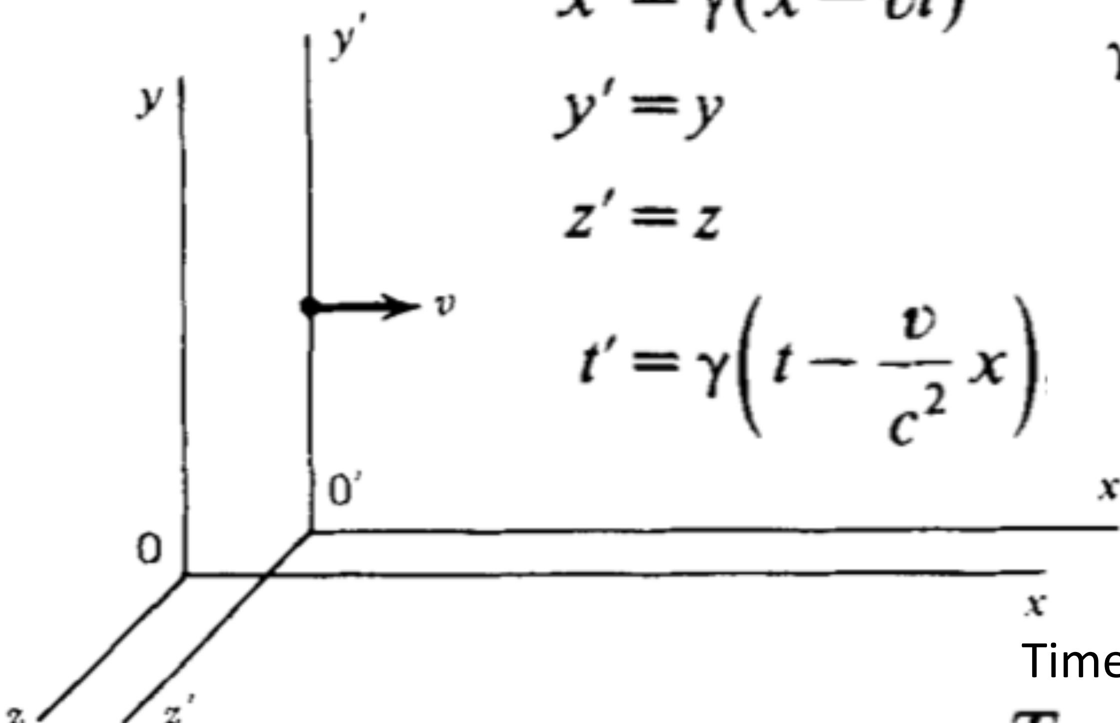
Frames K and K' with relative velocity v along x axis

A pulse of light emitted at t=0
Each observer will see an expanding sphere centered on its origin

Equations of expanding sphere

$$x^2+y^2+z^2-c^2t^2=0, \qquad x'^2+y'^2+z'^2-c^2t'^2=0,$$

Review of Lorentz Transformations



$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

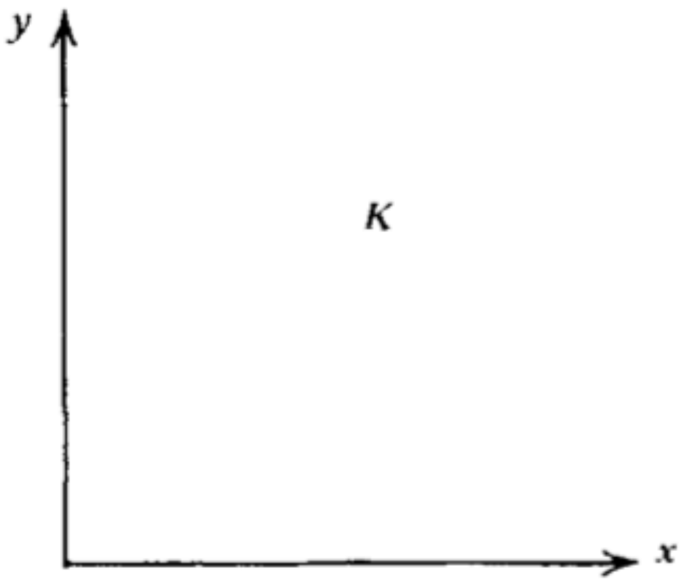
Length contraction

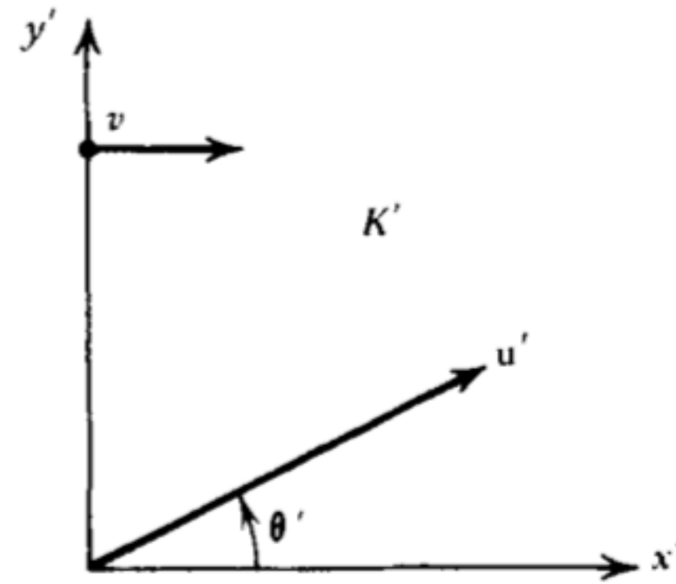
$$L = \left(1 - \frac{v^2}{c^2}\right)^{1/2} L_0$$

Time dilation

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T_0$$

Transformations of velocities

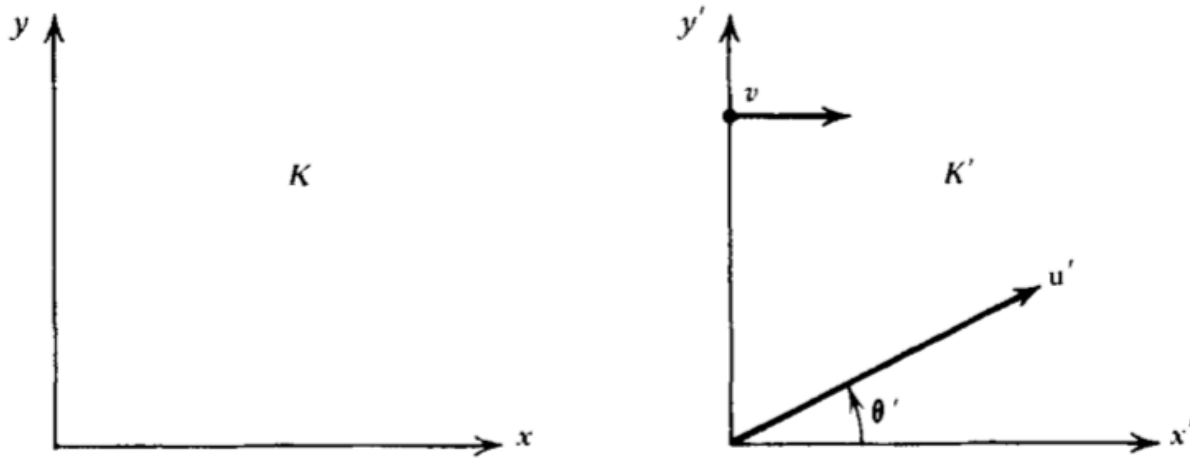




$$dx = \gamma(dx' + v\,dt'), \qquad dy = dy'$$

$$dz = dz', \qquad dt = \gamma\left(dt' + \frac{v}{c^2}\,dx'\right)$$

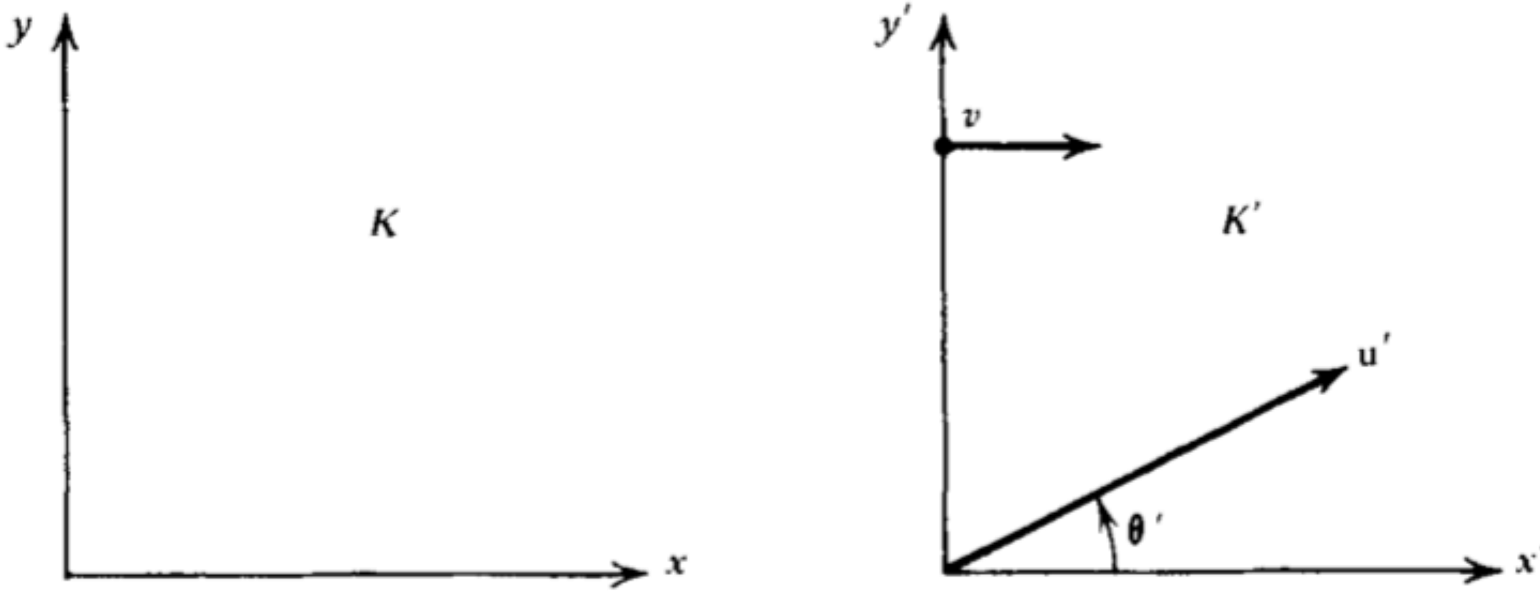
Transformations of velocities



$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{u'_x + v}{1 + vu'_x/c^2}$$

$$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \qquad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}$$

Transformations of velocities



$$u_{\|} = \frac{u'_{\|} + v}{\left(1 + v u'_{\|}/c^2\right)}, \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + v u'_{\|}/c^2\right)}$$

Components of **u** parallel and perpendicular to **v**

$$\tan\theta = \frac{u_{\perp}}{u_{\|}} = \frac{u' \sin\theta'}{\gamma(u' \cos\theta' + v)},$$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + v/c)}$$

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + (v/c)\cos\theta'}$$

Transformations of velocities
Beaming effect

For θ' = π/2, considering a photon emitted at right angles to v in K' →

$$\tan\theta = \frac{c}{\gamma v}$$

$$\sin\theta = \frac{1}{\gamma}.$$

For highly relativistic speeds γ >>1 →

$$\theta \sim \frac{1}{\gamma}$$

Consider photons are emitted isotropically in K'.
Half will have θ' >π/2 and other half will have θ' <π/2

In frame K the photons are concentrated in forward direction in a cone of 1/γ.
This is called **beaming effect**.

Transformations of velocities

Beaming effect

For θ′ = π/2, considering a photon emitted at right angles to v in K′ →

$$\tan\theta = \frac{c}{\gamma v}$$

$$\sin\theta = \frac{1}{\gamma}.$$

For highly relativistic speeds γ >>1 →

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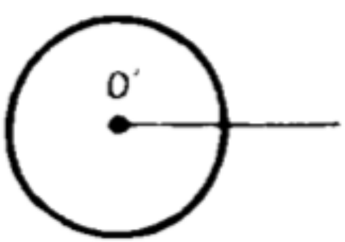


Isotropic emission: Rest frame K′

Beamed emission :K

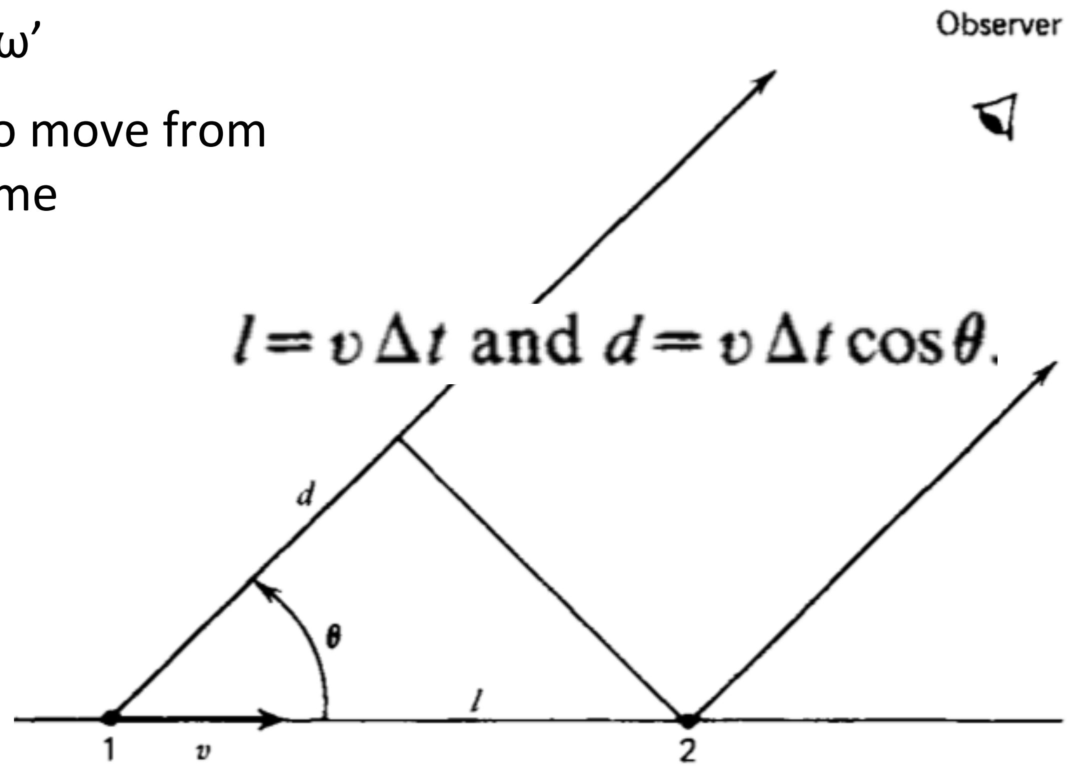
Doppler effect

Consider in rest frame of K
a source emits one period of radiation as it moves from point 1 to point 2

Rest frame frequency of radiation ω'

Time dilation implies, time taken to move from point 1 to point 2 in observer's frame

$$\Delta t = \frac{2\pi\gamma}{\omega'}$$



$$l = v\,\Delta t \text{ and } d = v\,\Delta t \cos\theta.$$

Difference in arrival time of the radiation emitted at 1 and 2

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right)$$

Observed frequency

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}$$

→ **Relativistic Doppler effect**

Proper time

Space and time have different values in different frames are separately subject to Lorentz transformation

Some quantities that are same in all Lorentz frames called Lorentz invariants

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Proper time $d\tau$ is unchanged under Lorentz transformation
Proper time measures time interval between events in same spatial location

Four vectors

One can find Lorentz transformation properties of other quantities as well. However four vectors have transformation properties identical to co-ordinates of events. So the treatment is less complicated.

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

↓

quantities x,y,z,t an be formed in to a vector in four-dimensional space

Define

$$x^0 = ct$$

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z.$$

Space-time is a four-vector: $x^\mu = [ct, x]$
For μ=0,1,2,3

Four vectors

Four vectors −Four components that transform in a specific way under Lorentz transformation

Length of Four vectors is invariant i.e. same in every inertial system

Electromagnetism predicts that waves travel at c in vacuum.

Laws of electro magnetism must be Lorentz invariant.

Special relativity in one slide

Space-time is a four-vector: $x^\mu = [ct, \mathbf{x}]$

Four-vectors have Lorentz transformations between two frames with uniform relative velocity v:

$$x' = \gamma(x - \beta ct); \qquad ct' = \gamma(ct - \beta x)$$

$$\beta = v/c \text{ and } \gamma = 1/\sqrt{1-\beta^2}$$

Lengths of four vectors are Lorentz invariant

$x^\mu x^\nu = c^2t^2 - |x|^2 = c^2t'^2 - |x'|^2 = s^2$

Charge and Current densities

Under a Lorentz transformation a static charge q at rest becomes a charge moving with velocity v. This is a current.

A static charge density ρ at one frame becomes a current density J in other

Note: Charge is conserved by a Lorentz transformation

The charge/current four-vector is:

$$J^{\mu} = \rho dx^{\mu}/dt = [c\rho, \mathbf{J}]$$

The full Lorentz transformation is:

$$J'_x = \gamma(J_x - v\rho); \qquad \rho' = \gamma(\rho - v/c^2\, J_x)$$

Note: γ factor can be understood as a length contraction or time dilation affecting the charge and current densities

Electrostatic & vector potentials

A static charge density ρ is a source of an electrostatic potential V
A current density J is a source of a magnetic vector potential A

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{r} d\tau \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}}{r} d\tau$$

Under a Lorentz transformation a V becomes an A:

$$A'_x = \gamma(A_x - \frac{v}{c^2}V) \qquad V' = \gamma(V - vA_x)$$

The potential four-vector is

$$A^\mu = \left[\frac{V}{c}, \mathbf{A}\right]$$

End of Lecture 6

Next lecture : 28th August