Electrodynamics and Radiative Processes I

Lecture 5 – Radiation from moving charges

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Recap Lecture 4

Maxwell's equations

Maxwell's equations in vacuum

Wave equation with **E**

Solution of wave equation with **E**

Recap Lecture 4

Maxwell’s equations

Maxwell’s equations in vacuum

Wave equation with **E**

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Solution of wave equation with **E** and **B**

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 \mathrm{e}^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

$$\mathbf{B} = \hat{\mathbf{a}}_2 B_0 \mathrm{e}^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

Maxwell’s Equations (Recap)

(in Gaussian units)

$$\nabla \cdot \mathbf{D} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \epsilon \mathbf{E},$$

$$\mathbf{B} = \mu \mathbf{H},$$

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

Electromagnetic Potentials

E and **B** are replaced by Φ(r,t) and **A(r,t)**

Why we need EM potentials?

1) One scalar plus one vector simpler than two vectors

2) Determining **A** and Φ are simpler

3) Relativistic EM theory will be simpler

Electromagnetic Potentials

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Maxwell's equation $\nabla \cdot \mathbf{B} = 0$

Vector potential **A(r,t)** defined as $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \Longrightarrow \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Electromagnetic Potentials

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2) Determining **A** and Φ are simpler

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Maxwell's equation $\nabla \cdot \mathbf{B} = 0$

Vector potential **A(r,t)** defined as $\mathbf{B} = \nabla \times \mathbf{A}.$

Thus

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \Longrightarrow \quad \nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

Scalar potential Φ(r,t) defined as $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$

Electromagnetic Potentials

E and B are replaced by Φ(r,t) and **A(r,t)**

$$\mathbf{E}+\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}=-\boldsymbol{\nabla}\phi$$

$$\Downarrow$$

$$\mathbf{E}=-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Remember

$$\boldsymbol{\nabla}\cdot\mathbf{D}=4\pi\rho$$

$$\boldsymbol{\nabla}^2\phi+\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A})=-4\pi\rho$$

Electromagnetic Potentials

Thus from Maxwell's equations,

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\downarrow \qquad\qquad \Downarrow \qquad\qquad \downarrow$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

Electromagnetic Potentials

$$\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$$

Thus from Maxwell’s equations,

$$\nabla\times(\nabla\times\mathbf{A})-\frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)=\frac{4\pi}{c}\mathbf{j}$$

$$\nabla\times(\nabla\times\mathbf{A})=-\nabla^2\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})$$

Electromagnetic Potentials

$$\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}$$

Thus from Maxwell's equations,

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↓

$$\nabla\times(\nabla\times\mathbf{A})=-\nabla^2\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})$$

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\cancel{\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}}\right)=-\frac{4\pi}{c}\mathbf{j}$$

Electromagnetic Potentials

Scalar and vector potential are not uniquely determined by the conditions

For example, the addition of gradient ψ to **A** will not change **B**

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\psi, \qquad \mathbf{B} \rightarrow \mathbf{B}.$$

Electric field will not change if ϕ is changed in following manner

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}, \qquad \mathbf{E} \rightarrow \mathbf{E}.$$

Such alterations of **A** and φ are called Gauge transformation

Lorentz Gauge	$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$

Electromagnetic Potentials

Thus from Maxwell's equations,

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -4\pi\rho$$

$$\Downarrow$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(\cancel{\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}}\right) = -4\pi\rho$$

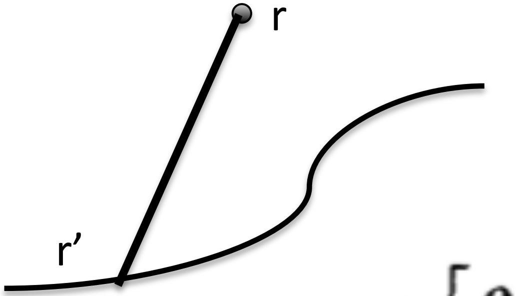
Thus from Maxwell's equations,

$$\nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = \frac{4\pi}{c}\mathbf{j}$$

$$\Downarrow$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\cancel{\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}}\right) = -\frac{4\pi}{c}\mathbf{j}$$

Retarded Potentials



$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \quad \longrightarrow \quad \phi(\mathbf{r},t) = \int \frac{[\rho]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \quad \longrightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{[\mathbf{j}]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$[Q] \equiv Q\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)$$

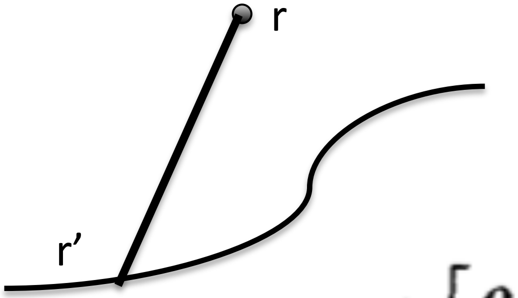
Information at point r’ propagates at speed of light.

↓

The potential at r can only be affected by conditions at r’ at a retarded time t- |r-r’|/c

Retarded Potentials

$$[Q] \equiv Q\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)$$



$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \Longrightarrow \phi(\mathbf{r}, t) = \int \frac{[\rho]d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},$$

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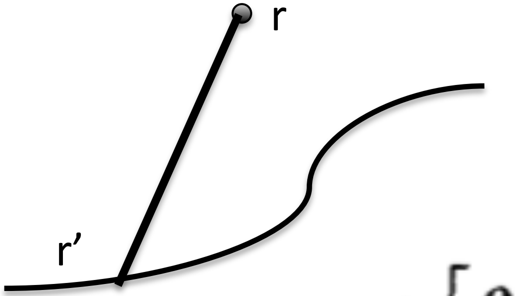
Retarded Potentials



$$[Q] \equiv Q\left(\mathbf{r}', t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho, \quad \longrightarrow \quad \phi(\mathbf{r}, t) = \int \frac{[\rho]\, d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},$$

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$$\boxed{\phi(\mathbf{r}, t) = \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\delta(t' - t + |\mathbf{r} - \mathbf{r}'|/c),}$$

Information at point r’ propagates at speed of light.

↓

The potential at r can only be affected by conditions at r’ at a retarded time t- |r-r’|/c

Retarded Potentials

✓ Retarded time refers to conditions at the point r’ that existed at a time earlier than t by time required for light to travel between r and r’

✓ Information from point r’ propagates at speed of light, so Potential at r can be affected by conditions of r’ at this retarded time

✓ Solutions with advanced time are not permitted physically.

For given charge and current density first find the retarded potentials and then determine **E** and **B**

Retarded potential of single moving charges : Lienard-Wiechart potentials

Particle of charge q moving along a trajectory $r=r_0(t)$, velocity $u(t)=\dot{r}_0(t)$

Charge and current density

$$\rho(\mathbf{r},t)=q\delta(\mathbf{r}-\mathbf{r}_0(t)),$$

$$\mathbf{j}(\mathbf{r},t)=q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

$$q=\int \rho(\mathbf{r},t)d^3\mathbf{r},$$

↓

Total Charge

$$q\mathbf{u}=\int \mathbf{j}(\mathbf{r},t)d^3\mathbf{r}.$$

↓

Total Current

Retarded potential of single moving charges : Lienard-Wiechart potentials

Particle of charge q moving along a trajectory $r=r_0(t)$, velocity $u(t)=\dot{r}_0(t)$

Charge and current density

$$\rho(\mathbf{r},t)=q\delta(\mathbf{r}-\mathbf{r}_0(t)),$$

$$\mathbf{j}(\mathbf{r},t)=q\mathbf{u}(t)\delta(\mathbf{r}-\mathbf{r}_0(t))$$

$$\boxed{\phi(\mathbf{r},t)=\int\frac{[\rho]\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}} \longrightarrow \text{Scalar Potential}$$

$$[Q]\equiv Q\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)$$

$$\phi(\mathbf{r},t)=\int d^3\mathbf{r}'\int dt'\,\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c),$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

Particle of charge q moving along a trajectory $r=r_0(t)$, velocity $u(t)=\dot{r}_0(t)$

Charge and current density

$$\rho(\mathbf{r},t)=q\delta(\mathbf{r}-\mathbf{r}_0(t)),$$

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$$[Q]\equiv Q\left(\mathbf{r}',t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)$$

$$\boxed{\phi(\mathbf{r},t)=\int\frac{[\rho]d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}} \longrightarrow$$ Scalar Potential

$$\phi(\mathbf{r},t)=\int d^3\mathbf{r}'\int dt'\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c),$$

$$\Big\downarrow \quad q=\int\rho(\mathbf{r},t)d^3\mathbf{r},$$

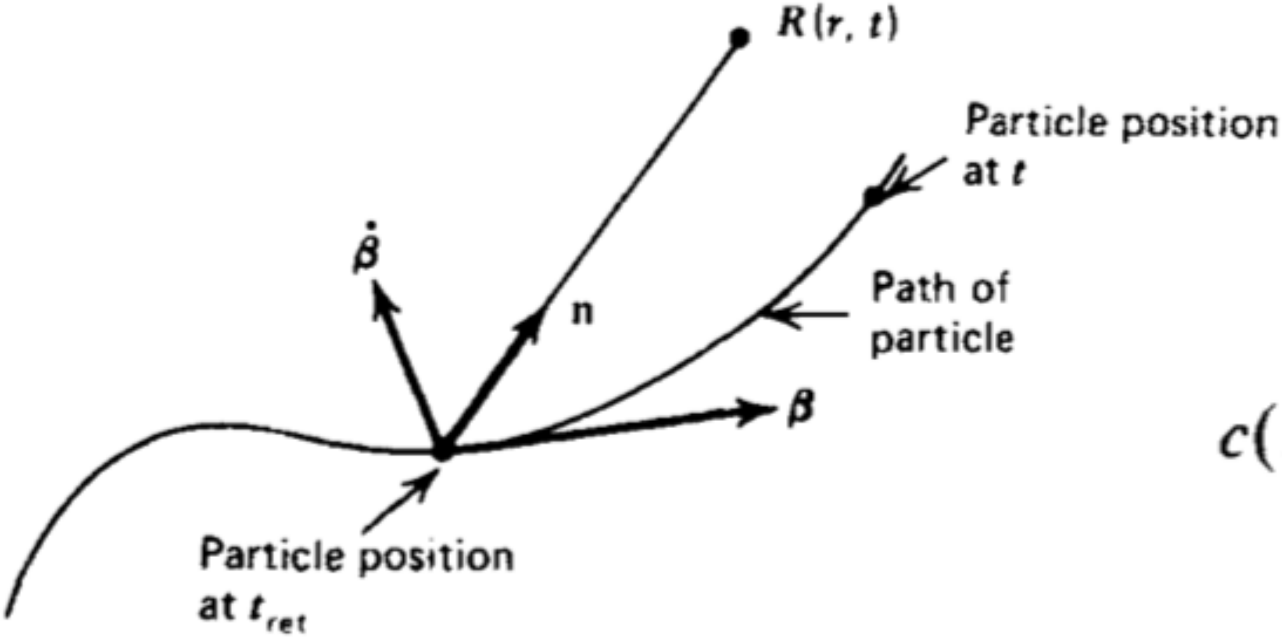
$$\phi(\mathbf{r},t)=q\int\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi(\mathbf{r},t)=q\int\delta(t'-t+|\mathbf{r}-\mathbf{r}_0(t')|/c)\frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|}$$

$$\mathbf{R}(t')=\mathbf{r}-\mathbf{r}_0(t')$$

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\delta\underline{(t'-t+R(t')/c)}dt'$$



δ function vanishes for
A value of $t'=t_{ret}$ given by,

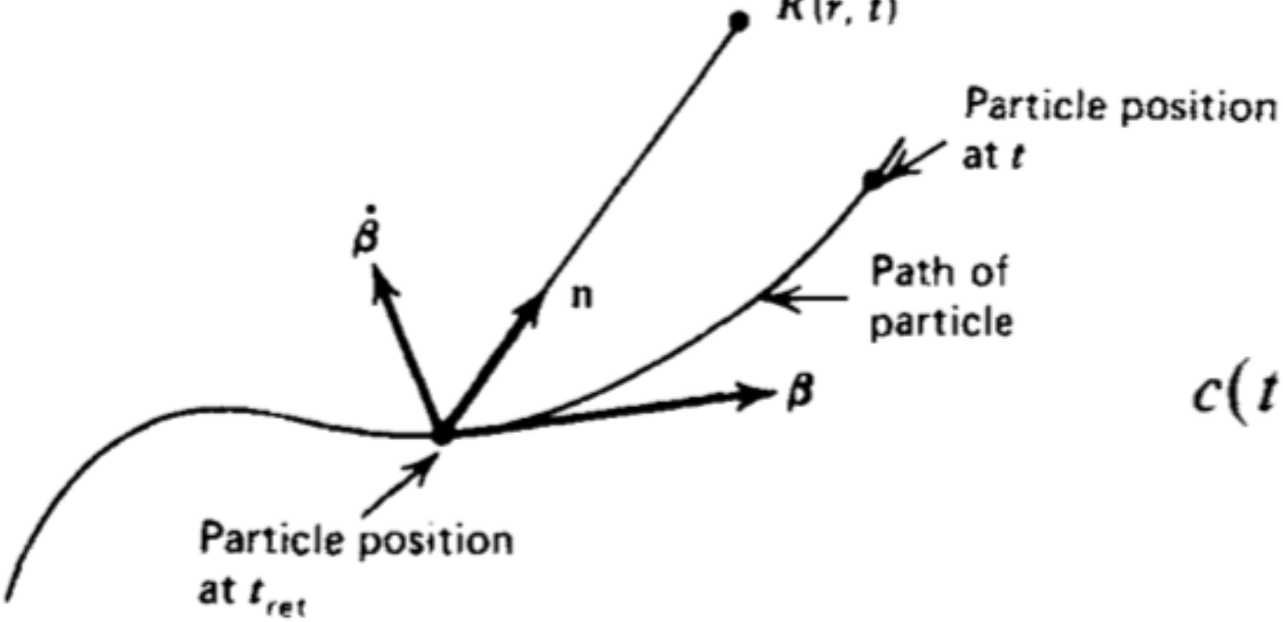
$$c(t-t_{\text{ret}})=R(t_{\text{ret}})$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\delta(t'-t+R(t')/c)dt'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{q}{c}\int \mathbf{u}(t')R^{-1}(t')\underline{\delta(t'-t+R(t')/c)}dt'$$

↓



δ function vanishes for
A value of $t'=t_{ret}$ given by,

$$c(t-t_{ret})=R(t_{ret}).$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

Change the variable from t' → t"

$$t'' = t' - t + [R(t')/c],\qquad \longrightarrow \qquad dt'' = dt' + \frac{1}{c}\dot{R}(t')\,dt'.$$

$$R^2(t') = \mathbf{R}^2(t')$$

$$2R(t')\dot{R}(t') = -2\mathbf{R}(t')\cdot\mathbf{u}(t'),$$

$$\mathbf{n} = \frac{\mathbf{R}}{R}$$

$$\Big\downarrow$$

$$dt'' = \left[1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\kappa(t') = 1 - \frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

Change the variable from t’ → t”

$$t''=t'-t+[R(t')/c] \longrightarrow dt''=dt'+\frac{1}{c}\dot{R}(t')dt'.$$

$$\Big\downarrow$$

$$dt''=\left[1-\frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]dt',$$

$$\phi(\mathbf{r},t)=q\int R^{-1}(t')\left[1-\frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')\right]^{-1}\delta(t'')dt''$$

$$\phi(\mathbf{r},t)=\frac{q}{\kappa(t_{\text{ret}})R(t_{\text{ret}})} \qquad \kappa(t')=1-\frac{1}{c}\mathbf{n}(t')\cdot\mathbf{u}(t')$$

Lienard-Wiechart Potential

$$\boxed{\phi=\left[\frac{q}{\kappa R}\right] \qquad \mathbf{A}=\left[\frac{q\mathbf{u}}{c\kappa R}\right].}$$

Retarded potential of single moving charges : Lienard-Wiechart potentials

$$\phi = \left[\frac{q}{\kappa R} \right] \qquad \mathbf{A} = \left[\frac{q\mathbf{u}}{c\kappa R} \right].$$

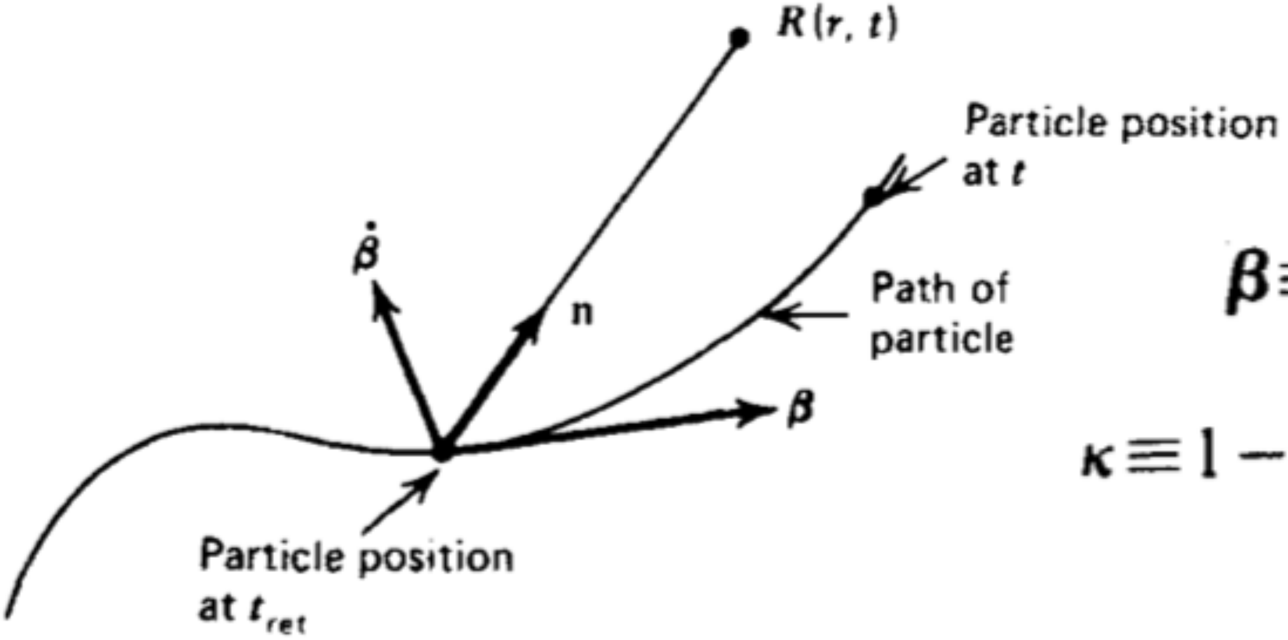
Differ from static electromagnetic theory in two ways

1) Extra factor κ : Important for velocities close to light.
   Tends to concentrate/beam potential into a narrow cone about particle velocity.
   Beaming effect (will be detailed in coming lectures)

2) Quantities are evaluated at retarded time.

Differentiate the potentials to get electric field(E) and magnetic field(B)
(Jackson Section 14)

Velocity and Radiation field



$$\boldsymbol{\beta} \equiv \frac{\mathbf{u}}{c}$$

$$\kappa \equiv 1 - \mathbf{n}\cdot\boldsymbol{\beta}.$$

Fig : Radiation field at R from position of the radiating particle at the retarded time

$$\boxed{\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]} \qquad \boxed{\mathbf{B}(\mathbf{r},t) = [\mathbf{n}\times\mathbf{E}(\mathbf{r},t)]}$$

Velocity field

Acceleration/Radiation field

Radiation field

$$\mathbf{E}(\mathbf{r},t)=q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3R^2}\right]+\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

Velocity field

- $1/R^2$ dependence
- Only contributing term for particle with constant velocity
- Generalization of the Coulomb's law to moving particles approaches to coulomb's law when u<<c

- Electric filed always point towards current position of the particle

Acceleration field/Radiation field

1/R dependence

Proportional to particle's acceleration perpendicular to n

Radiation field

$$\boxed{\mathbf{E}(\mathbf{r},t)=q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3R^2}\right]+\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]}$$

Radiation field

$$\mathbf{E}_{\text{rad}}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

$$\mathbf{B}_{\text{rad}}(\mathbf{r},t)=[\mathbf{n}\times\mathbf{E}_{\text{rad}}]$$

E_{rad}, B_{rad}, n : mutually perpendicular

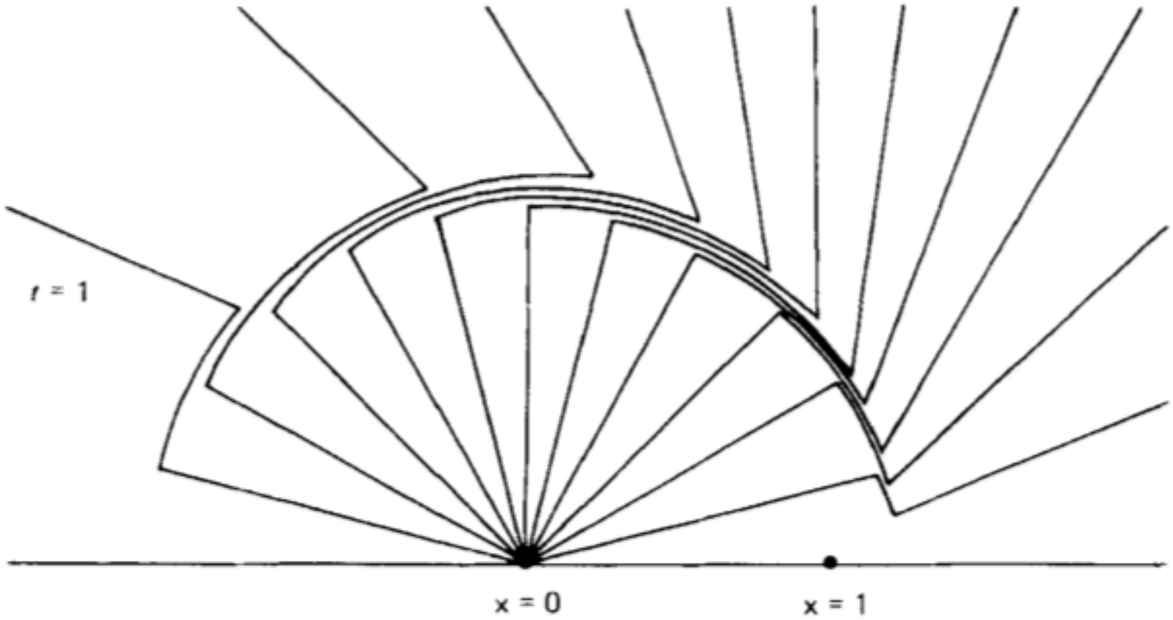
$|E_{rad}| = |B_{rad}|$

Radiation fields

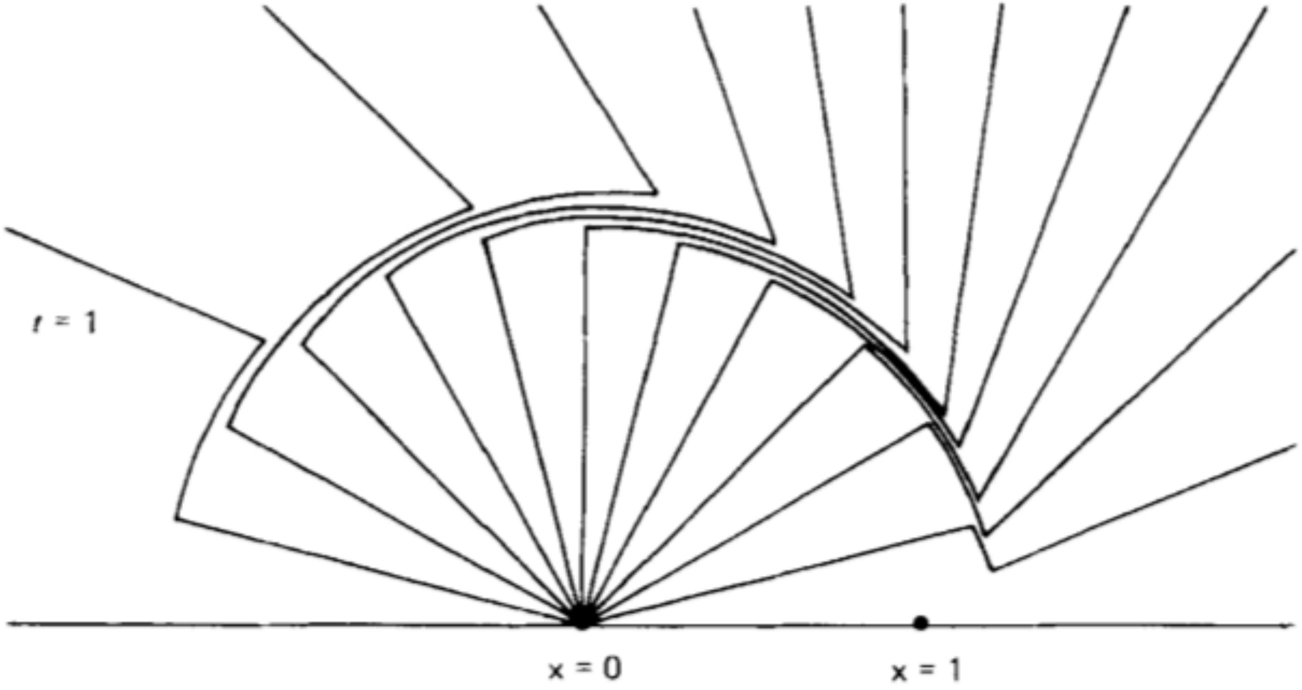
Consider a particle originally moving at constant velocity along x axis is stopped at x=0 and t=0

At t=1 the field outside of a radius c is radial and points to the position where particle would have been if there was no deceleration (since no information Is yet propagated to that distance)

But field inside the radius c is informed.



Radiation fields



These two fields can be connected with flux conservation: as shown in the figure.

Transition zone whose radial thickness is the time interval over which deceleration occurs.
This transition zone is almost transverse and much stronger.

Radius of the ring varies as R, strength of the field varies as 1/R

Radiation Spectrum

Energy per unit frequency per unit solid angle corresponding to the radiation field of a single particle

$$\frac{dW}{d\omega\, d\Omega} = \frac{c}{4\pi^2}\left|\int [R\mathbf{E}(t)]e^{i\omega t}dt\right|^2$$

(Slide 24 of Lecture 4)

$$= \frac{q^2}{4\pi^2 c}\left|\int \Big[\underbrace{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}}_{\downarrow}\Big]e^{i\omega t}dt\right|^2$$

Evaluated at a retarded time

Radiation Spectrum

Energy per unit frequency per unit solid angle corresponding to the radiation field of a single particle

$$\frac{dW}{d\omega\, d\Omega} = \frac{c}{4\pi^2}\left|\int [\,R\mathbf{E}(t)\,]e^{i\omega t}dt\right|^2$$

$$= \frac{q^2}{4\pi^2 c}\left|\int \left[\underline{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-3}}\right]e^{i\omega t}dt\right|^2$$

Evaluated at a retarded time

Changing variable from t to t’

$$t' = t - R(t')/c \qquad R(t') \approx |\mathbf{r}| - \mathbf{n}\cdot\mathbf{r}_0$$

$$dt = \kappa\, dt'$$

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2}\exp[\,i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)\,]dt'\right|^2.$$

Radiation fields

Energy per unit frequency per unit solid angle corresponding to the radiation field of a single particle

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int \mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\kappa^{-2}\exp\left[\,i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)\,\right]dt'\right|^2.$$

$$\Big\downarrow$$

Integration by parts

$$\frac{dW}{d\omega\, d\Omega} = (q^2\omega^2/4\pi^2 c)\left|\int \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\exp\left[\,i\omega(t'-\mathbf{n}\cdot\mathbf{r}_0(t')/c)\,\right]dt'\right|^2.$$

Radiation from non-relativistic systems of particles

Electric filed of moving charges (Refer to Slide 26)

$$\boxed{\mathbf{E}(\mathbf{r},t)=q\underbrace{\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^{2})}{\kappa^{3}R^{2}}\right]}_{\mathbf{E}_{\mathbf{vel}}}+\underbrace{\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^{3}R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]}_{\mathbf{E}_{\mathbf{rad}}}}$$

Knowing the velocity and radiation fields we will be able to discuss many radiation processes involving moving charges

For the moment we will consider discussion of non relativistic particles

$$|\beta|=\frac{u}{c}\ll 1 \quad\Longrightarrow\quad \frac{E_{\mathrm{rad}}}{E_{\mathrm{vel}}}\sim\frac{R\dot{u}}{c^{2}}$$

Radiation from non-relativistic systems of particles

Refer to Slide 26

$$\boxed{\mathbf{E}(\mathbf{r},t)=q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3R^2}\right]+\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]}$$

Considering $|\beta|=\frac{u}{c}\ll 1$ $\qquad \frac{E_{\mathrm{rad}}}{E_{\mathrm{vel}}}\sim\frac{R\dot{u}}{c^2}$

For particle with frequency of oscillation ν $\qquad \frac{E_{\mathrm{rad}}}{E_{\mathrm{vel}}}\sim\frac{Ru\nu}{c^2}=\frac{u}{c}\frac{R}{\lambda}.$

$R < \lambda$ → "Near zone" → Velocity field stronger than Radiation field by > c/u

$R \gg \lambda(c/u)$ → "Far zone" → Acceleration field dominates Domination increase linearly with R

Larmor's Formula

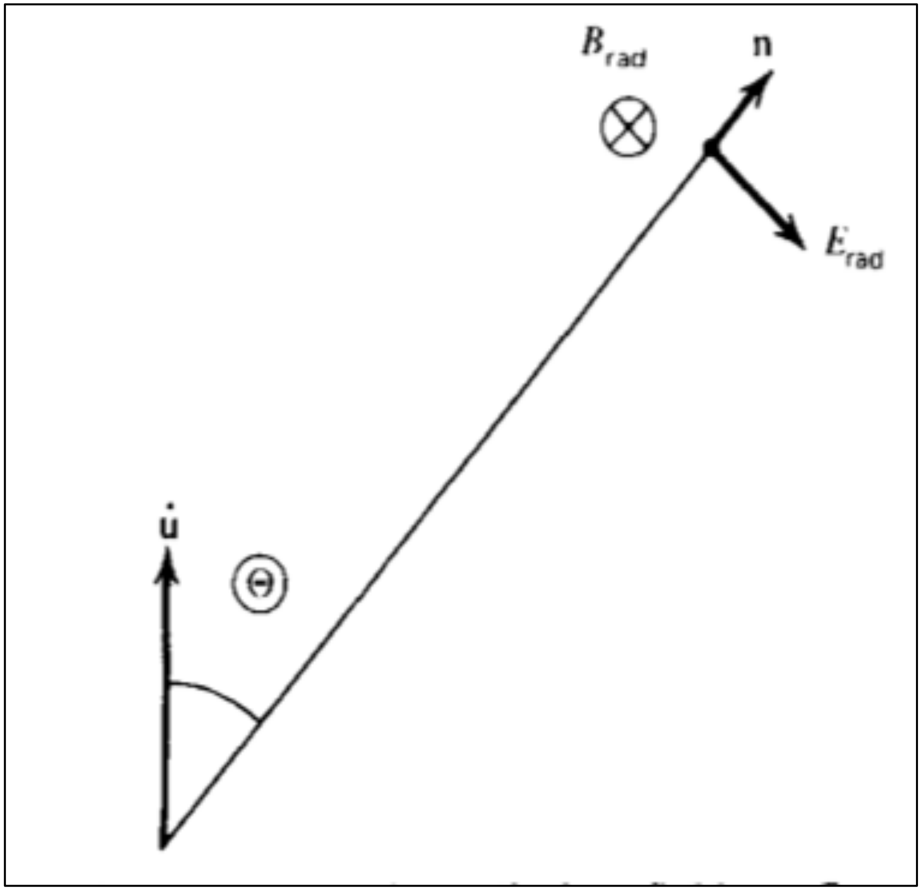
Total power radiated by a non-relativistic point charge as it accelerates

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

For β<<1

$$\mathbf{E}_{\mathrm{rad}}=\left[(q/Rc^2)\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]$$

$$\mathbf{B}_{\mathrm{rad}}=\left[\mathbf{n}\times\mathbf{E}_{\mathrm{rad}}\right]$$



Outward flow of energy along **n**

Larmor's Formula

Total power radiated by a non-relativistic point charge as it accelerates

$$\mathbf{E}_{\text{rad}}(\mathbf{r},t)=\frac{q}{c}\left[\frac{\mathbf{n}}{\kappa^3 R}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}\right]$$

For β<<1

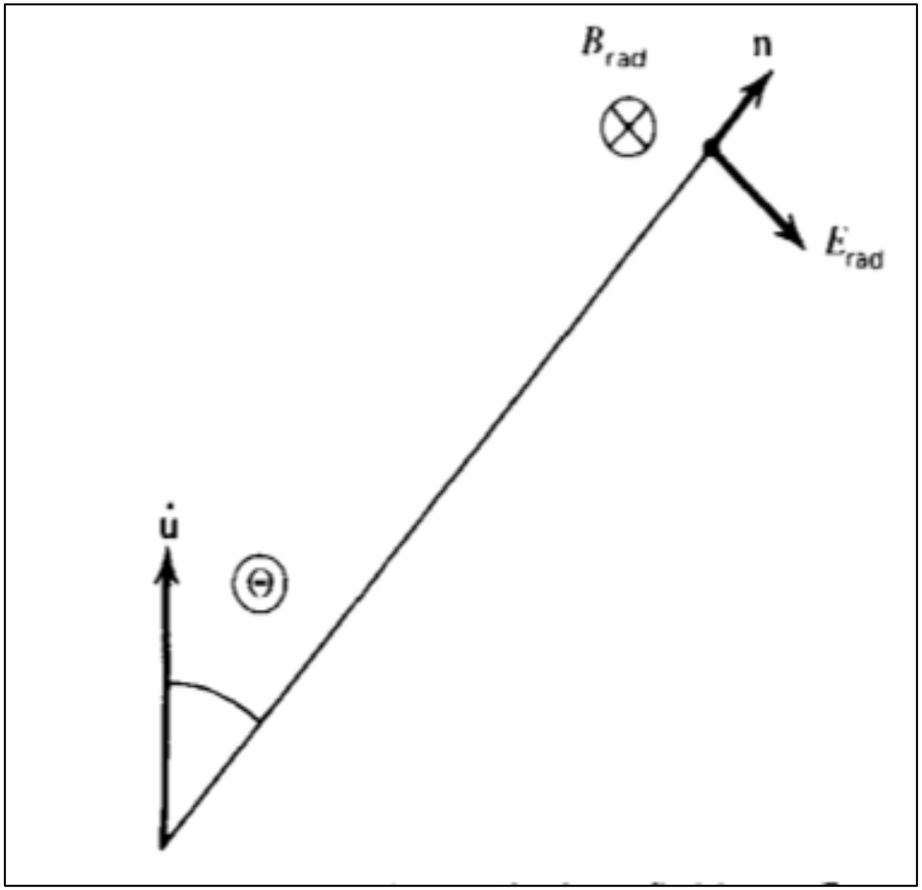
$$\mathbf{E}_{\text{rad}}=\left[(q/Rc^2)\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}})\right]$$

$$\mathbf{B}_{\text{rad}}=\left[\mathbf{n}\times\mathbf{E}_{\text{rad}}\right].$$

$$|\mathbf{E}_{\text{rad}}|=|\mathbf{B}_{\text{rad}}|=\frac{q\dot{u}}{Rc^2}\sin\Theta$$

Poynting Vector

$$S=\frac{c}{4\pi}E_{\text{rad}}^2=\frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta$$



Outward flow of energy along **n**

Larmor’s Formula

$$S=\frac{c}{4\pi}E_{\mathrm{rad}}^2=\frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta$$

Power radiated per unit solid angle per unit time

$$\frac{dW}{dt\,d\Omega}=\frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta. \quad \longrightarrow \quad P=\frac{dW}{dt}=\frac{q^2\dot{u}^2}{4\pi c^3}\int\sin^2\Theta\,d\Omega$$

$$\boxed{P=\frac{2q^2\dot{u}^2}{3c^3}}$$

↓

Larmor’s Formula for emission
from a single accelerated charge q

Larmor’s Formula

$$\boxed{P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int \sin^2 \Theta \, d\Omega}$$

$$\boxed{P = \frac{2 q^2 \dot{u}^2}{3 c^3}}$$

✓ Power emitted is proportional to square of charge and square of acceleration

✓ Dependence on $\sin^2 \Theta$: No radiation along direction of acceleration
Max radiation perpendicular to acceleration

✓ Direction of E_{rad} is determined by **ů** and **n** : If the particle accelerates along a line radiation will be 100% polarized in the plane of **ů** and **n**

Larmor's Formula

$$\boxed{P = \frac{2q^2\dot{u}^2}{3c^3}}$$

- ✓ Larmor Formula states that any charged particle radiates when accelerated and that the total radiated power is proportional to the square of the acceleration.

- ✓ Radiation are only emitted when particle are accelerated.

- ✓ Since the astrophysical accelerations are usually electromagnetic, the acceleration is usually proportional to the charge/mass ratio of the particle.

- ✓ Radiation from electrons is typically ~10^6 stronger than radiation from protons, which are ~10^3 times more massive.

Larmor's equation will be applied in many contexts
e.g. dipole approximation as well as for free-free
synchrotron emission from astrophysical sources

Larmor’s Formula

$$\boxed{P = \frac{2q^2\dot{u}^2}{3c^3}}$$

Limitations:

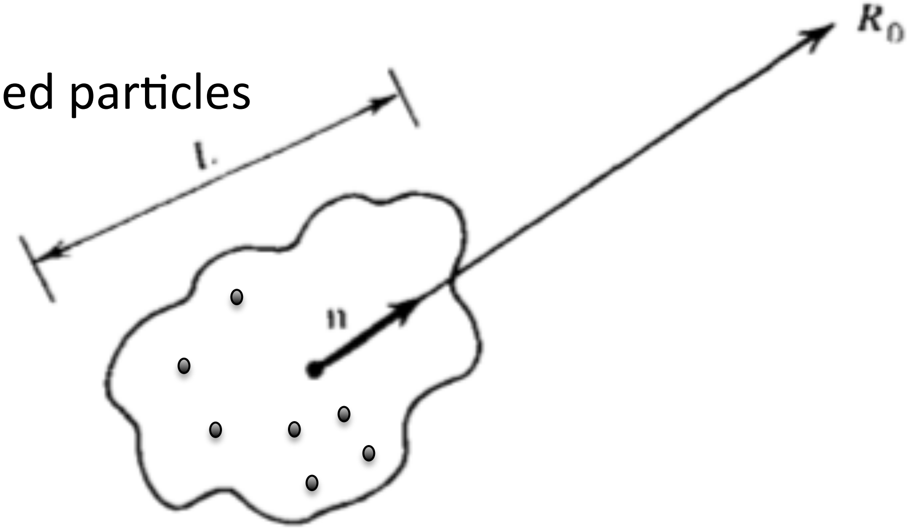
- ✓ Larmor's formula is nonrelativistic; it is valid only in frames moving at velocities v<<c with respect to the radiating particle.

- ✓ To treat particles moving at nearly the speed of light in the observer's frame, we must use Larmor's equation to calculate the radiation in the particle's rest frame and then transform the result to the observer's frame in a relativistically correct way.

- ✓ Larmor's formula does not incorporate the constraints of quantum mechanics, so it should be applied with great caution to microscopic systems such as atoms. For example, Larmor's equation incorrectly predicts that the electron in a hydrogen atom will quickly radiate away all of its kinetic energy and fall into the nucleus.

Dipole approximation

Collection of charged particles



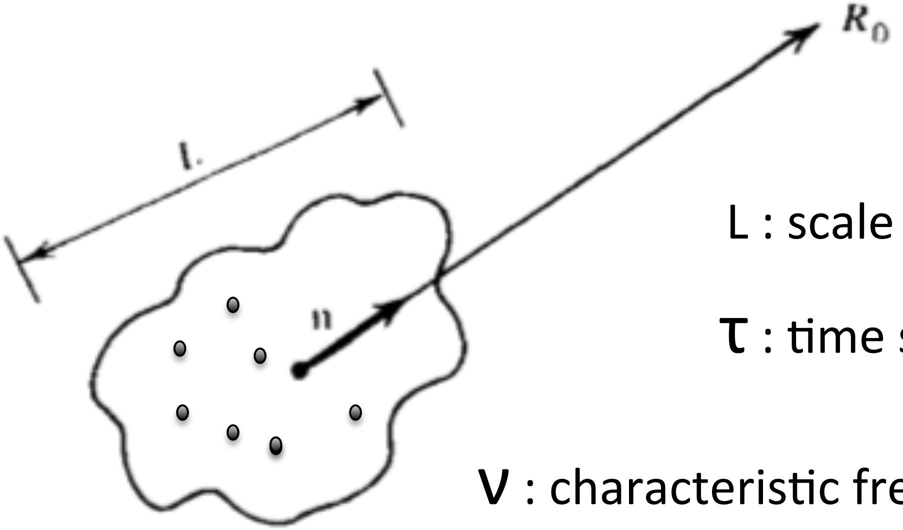
When there are many particles with position $r_{i,}$ velocities u_i, charges q_i

Radiation field at large distance ~ summation of E_{rad} for each particle

But E_{rad} for each particle is true for different retarded times

How to derive radiation field?

Dipole approximation



L : scale of the system

τ : time scale for changes

ν : characteristic frequency of E_{rad}=1/τ

Differences in retarded time across source is negligible ⟺ $\tau \gg L/c$

$$\frac{c}{\nu} \gg L$$

Differences in retarded time can be ignored if
size of the system is small compared to wavelength ⟺ $\lambda \gg L$

Dipole approximation

$$\begin{tikzcd}
\mathbf{E}_{\text{rad}} = \sum_i \frac{q_i}{c^2} \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{u}}_i)}{R_i} \arrow[r, "\mathbf{d} = \sum_i q_i \mathbf{r}_i."] & \mathbf{E}_{\text{rad}} = \frac{\mathbf{n} \times (\mathbf{n} \times \ddot{\mathbf{d}})}{c^2 R_0} \\
\frac{dP}{d\Omega} = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} \sin^2 \Theta, \arrow[r] & \boxed{P = \frac{2\ddot{\mathbf{d}}^2}{3c^3}} \arrow[d] \\
 & {}
\end{tikzcd}$$

Dipole approximation :
Larmor’s formula extended for a collection of non-relativistic particles

Dipole approximation

Spectrum of radiation $\frac{dW}{d\omega}$

Assuming **d** lies in single direction

$$E(t)=\ddot{d}(t)\frac{\sin\Theta}{c^2R_0}$$

$$\hat{E}(\omega)=-\frac{1}{c^2R_0}\omega^2\hat{d}(\omega)\sin\Theta$$

Electric field in frequency domain

Fourier transform of d(t)

$$d(t)=\int_{-\infty}^{\infty}e^{-i\omega t}\hat{d}(\omega)d\omega.$$

$$\ddot{d}(t)=-\int_{-\infty}^{\infty}\omega^2\hat{d}(\omega)e^{-i\omega t}d\omega.$$

Dipole approximation

Spectrum of radiation $\frac{dW}{d\omega}$

Assuming **d** lies in single direction

Electric field in time domain →

$$E(t)=\ddot{d}(t)\frac{\sin\Theta}{c^2R_0}$$

Electric field in frequency domain →

$$\hat{E}(\omega)=-\frac{1}{c^2R_0}\omega^2\hat{d}(\omega)\sin\Theta$$

From Lecture 4, energy per unit area

$$\frac{dW}{dA}=c\int_0^\infty|\hat{E}(\omega)|^2d\omega,$$

Energy per unit solid angle per frequency range, (dA= R_0^2 dΩ)

$$\frac{dW}{d\omega\, d\Omega}=\frac{1}{c^3}\omega^4|\hat{d}(\omega)|^2\sin^2\Theta.$$

Spectrum of radiation →

$$\frac{dW}{d\omega}=\frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2$$

Dipole approximation

Spectrum of radiation

Spectrum of radiation

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2$$

Rayleigh scattering formula
proportional to $1/\lambda^4$
(Reason for blue color of the sky)

Dipole approximation

Spectrum of radiation

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2$$

↓

Spectrum of the emitted radiation is related to frequency of oscillation of the dipole moments.

This property is not true for particles with relativistic velocities.

General multipole expansion

Vector Potential

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\int d^3\mathbf{r}'\int dt'\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c),$$

But Fourier transform of **A(r,t)** and **j**$_\omega$ are following

$$\mathbf{A}_\omega(\mathbf{r})=\int \mathbf{A}(\mathbf{r},t)e^{i\omega t}dt.$$

$$\mathbf{j}_\omega(\mathbf{r})=\int \mathbf{j}(\mathbf{r},t)e^{i\omega t}dt,$$

Thus Vector Potential

$$\mathbf{A}_\omega(\mathbf{r})=\frac{1}{c}\int\frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}',$$

General multipole expansion

Let us choose origin of the co-ordinates inside source of size L

$$\mathbf{A}_\omega(\mathbf{r}) = \int \mathbf{A}(\mathbf{r}, t) e^{i\omega t} dt.$$

Then for field s r>>L, we have

$$|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{n} \cdot \mathbf{r}'$$

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{1}{c} \int \frac{\mathbf{j}_\omega(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{ik|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$

n points towards the field point r and where r=|**r**|

$$\mathbf{A}_\omega(\mathbf{r}) \approx (\underline{e^{ikr}/cr}) \int \mathbf{j}_\omega(\mathbf{r}') \underline{e^{-ik\mathbf{n}\cdot\mathbf{r}'}} d^3\mathbf{r}'.$$

Effect of retardation from the source as a whole

Relative retardation of each element of the source

General multipole expansion

$$\mathbf{A}_\omega(\mathbf{r}) \approx (e^{ikr}/cr)\int \mathbf{j}_\omega(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}'.$$

Expanding the exponential in the integral considering kL<<1,

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{e^{ikr}}{cr}\sum_{n=0}^{\infty}\frac{1}{n!}\int \mathbf{j}_\omega(\mathbf{r}')(-ik\mathbf{n}\cdot\mathbf{r}')^n d^3\mathbf{r}'$$

Dipole approximation results from taking only the first term of the expansion (n=0)

$$\mathbf{A}_\omega(\mathbf{r})\Big|_{\text{dipole}} = \frac{e^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')d^3\mathbf{r}'$$

Quadrupole term is the term with n=1

$$\mathbf{A}_\omega(\mathbf{r})\Big|_{\text{quad}} = \frac{-ike^{ikr}}{cr}\int \mathbf{j}_\omega(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')d^3\mathbf{r}'$$

End of Lecture 5

Mini-test on lectures 1-6
(practice problems from Rybicki and Lightman Chapter 1, 2, 3)
: 28rd August

Next lecture : 23rd August