Electrodynamics and Radiative Processes I

Lecture 4 – Basic theory of radiation fields

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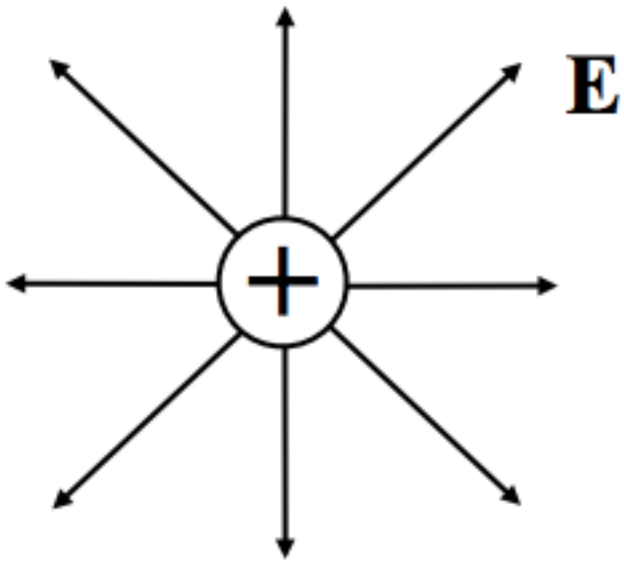
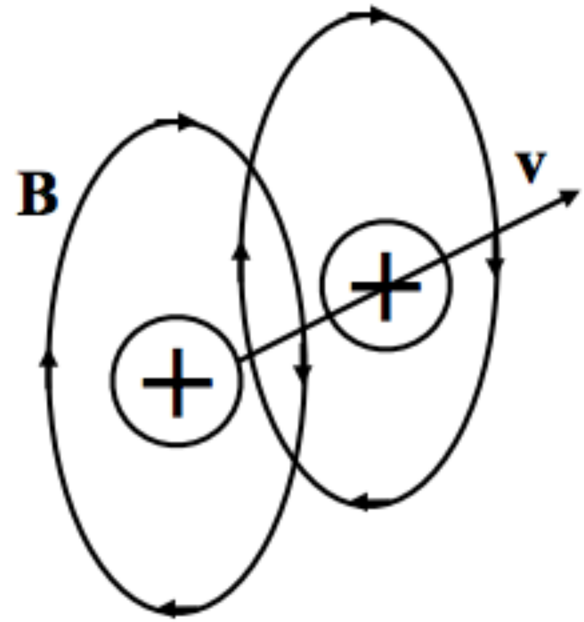
Electric and magnetic field





Static charge → Electric field

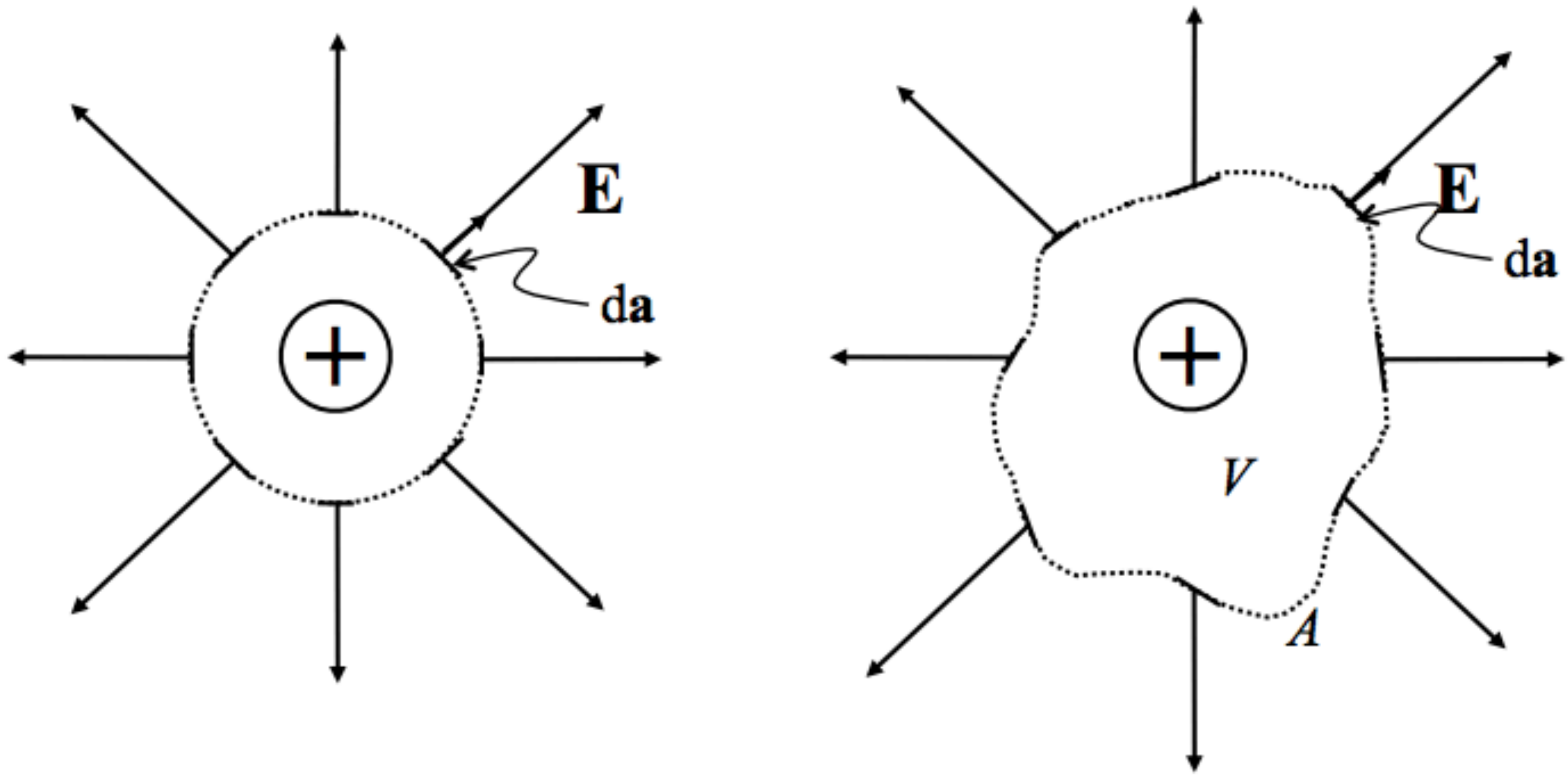
Moving charge → Magnetic field
→ Electric current

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Unit of electric field ?

Unit of Magnetic field ?

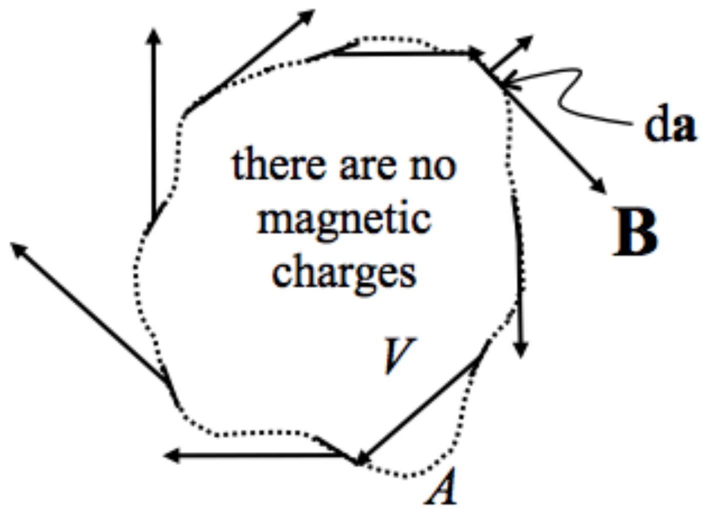
Gauss law: Electric field



$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho \mathrm{d}V \quad \xleftrightarrow{\text{Gauss theorem}} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Flux of E through a closed surface = (charge inside)/ε_0

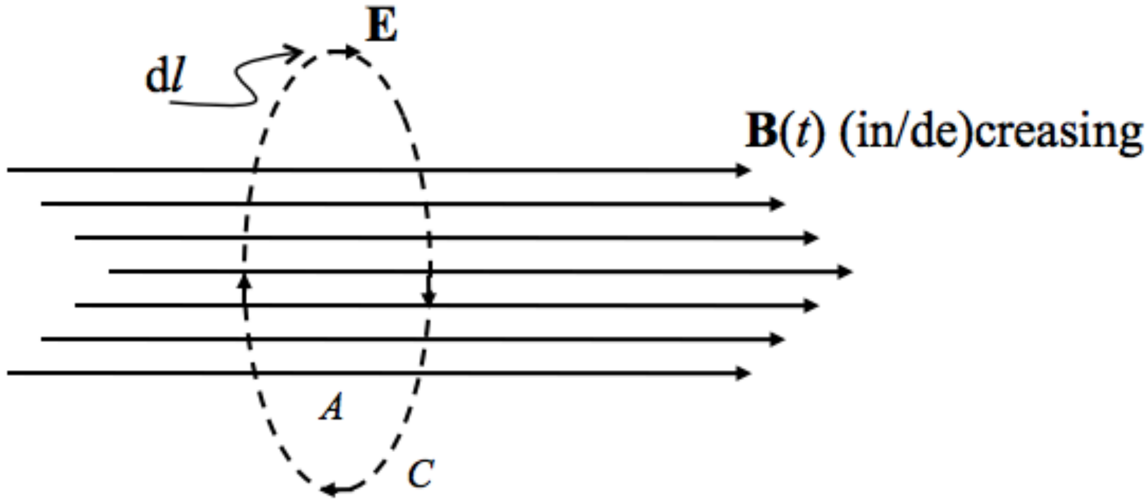
Gauss law : Magnetic field



$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0 \quad \overset{\text{Gauss theorem}}{\Longleftrightarrow} \quad \nabla \cdot \mathbf{B} = 0$$

Flux of B through a closed surface = 0

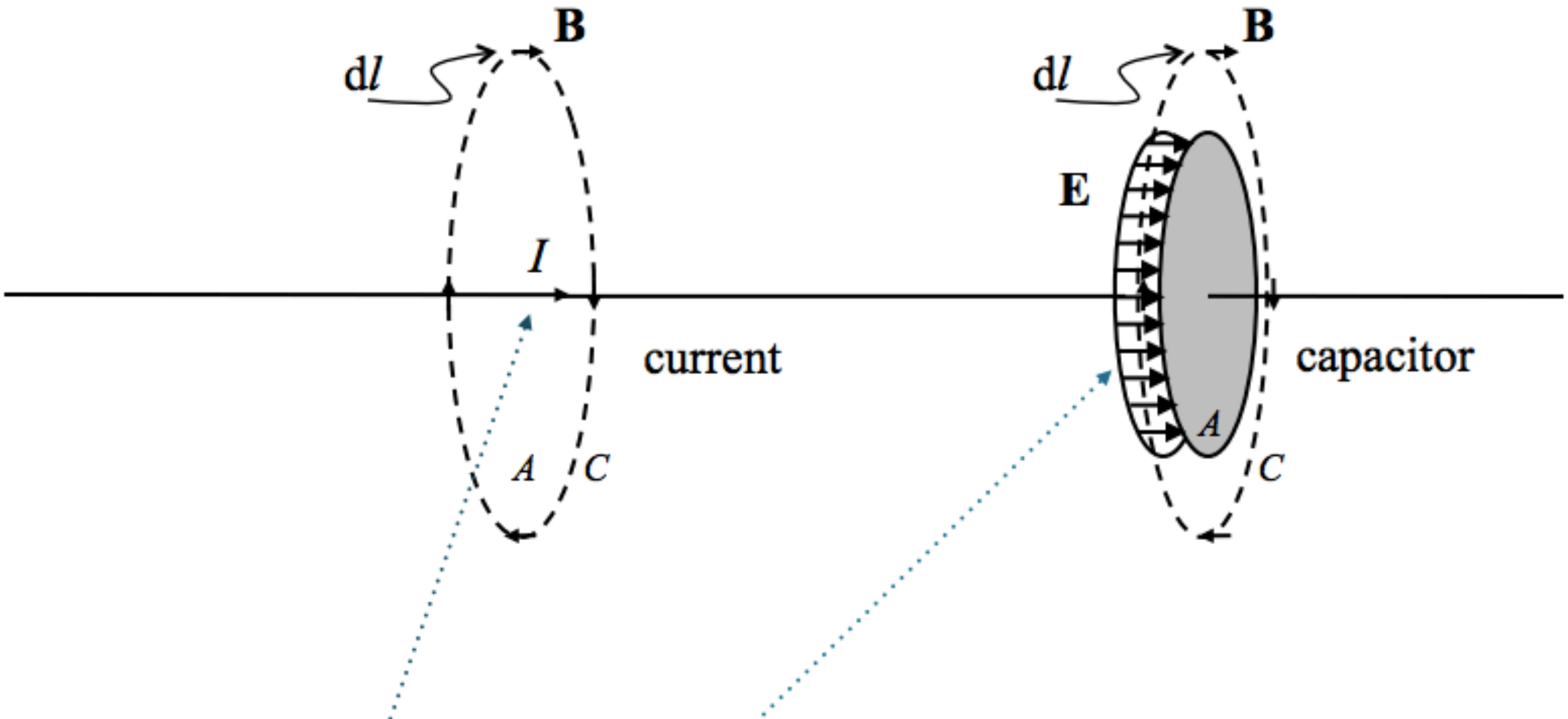
Faraday's law



$$\oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} \quad \overset{\text{Stokes theorem}}{\Longleftrightarrow} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Line integral of E around a loop = - d/dt (Flux of B through the loop)

Ampere-Maxwell's law: Magnetic induction



$$\oint_C \mathbf{B} \cdot \mathrm{d}l = \mu_0 \left(\iint_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \epsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} \right) \xLeftrightarrow{\text{Stokes theorem}} \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Integral of B around a loop = (Current through the loop)

+ε_0 d/dt(Flux of E through the loop)

Maxwell's Equations
(in SI units)

Integral form

$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho \mathrm{d}V$$

$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$$

$$\oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

$$\oint_C \mathbf{B} \cdot \mathrm{d}l = \mu_0 \left(\iint_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \epsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} \right)$$

Differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Electromagnetic flux

Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

Electromagnetic flux

Lorentz force $$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

Rate of work done $$\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \mathbf{E},$$

Current density $$\mathbf{j} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i} q_i \mathbf{v}_i,$$

Electromagnetic flux

Lorentz force
$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$

Rate of work done
$$\mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot \mathbf{E},$$

Current density
$$\mathbf{j} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i,$$

Rate of work done
$$\frac{1}{\Delta V} \sum_i q_i \mathbf{v}_i \cdot \mathbf{E} = \mathbf{j} \cdot \mathbf{E}.$$

Rate of change in mechanical energy of system per unit volume
$$\frac{dU_{\text{mech}}}{dt} = \mathbf{j} \cdot \mathbf{E}.$$

Maxwell's Equations
(in Gaussian units)

$$\nabla \cdot \mathbf{D} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's Equations
(in Gaussian units)

$$\nabla \cdot \mathbf{D} = 4\pi\rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \epsilon \mathbf{E},$$

$$\mathbf{B} = \mu \mathbf{H},$$

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

Electromagnetic flux

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]$$

Using the following relation

$$\mathbf{E}\cdot(\nabla\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\nabla\cdot(\mathbf{E}\times\mathbf{H}),$$

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}-c\nabla\cdot(\mathbf{E}\times\mathbf{H})-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]$$

Electromagnetic flux

$$\mathbf{j}\cdot\mathbf{E}=\frac{1}{4\pi}\left[c(\nabla\times\mathbf{H})\cdot\mathbf{E}-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right]$$

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Poynting's theorem

$$\underbrace{\mathbf{j}\cdot\mathbf{E}}_{\text{Rate of change of Mechanical energy}}+\underbrace{\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2+\frac{B^2}{\mu}\right)}_{\text{Rate of change of field energy } U_E+U_B}=\underbrace{-\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)}_{\text{Divergence of field energy flux}}.$$

Electromagnetic flux

Electromagnetic field energy per unit volume

$$U_{\text{field}} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{B^2}{\mu}\right) = U_E + U_B$$

Electromagnetic flux vector or Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.$$

Electromagnetic flux $\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}.$

Integrating over a volume element we get

$$\int_V \mathbf{j}\cdot\mathbf{E}\,dV + \frac{d}{dt}\int_V \frac{\epsilon E^2 + B^2/\mu}{8\pi}\,dV = -\int_\Sigma \mathbf{S}\cdot d\mathbf{A},$$

$$\frac{d}{dt}(U_{\text{mech}} + U_{\text{field}}) = -\int_\Sigma \mathbf{S}\cdot d\mathbf{A}$$

Rate of change of total (mechanical plus field) energy with in the volume V is equal to the net inward flow of energy through the bounding surface.

Plane electromagnetic waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

How ?

Wave equation with B ?

Plane electromagnetic waves

Maxwell's equation in vacuum

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

wave vector

frequency

Solutions of wave equation

$$\mathbf{E} = \boxed{\hat{\mathbf{a}}_1} \boxed{E_0}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\mathbf{B} = \boxed{\hat{\mathbf{a}}_2} \boxed{B_0}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

unit vectors

complex constants

Plane electromagnetic waves

Substitution in Maxwell's equation

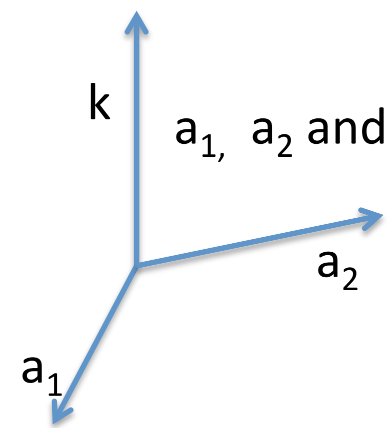
$$i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0 = 0 \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 = 0$$

a_1 and a_2 are perpendicular to k

$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 = \frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 = -\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0.$$

a_1 and a_2 are perpendicular

a_1, a_2 and k are perpendicular



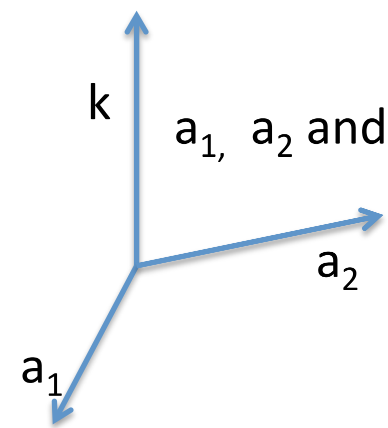
Plane electromagnetic waves

Substitution in Maxwell's equation

$$i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0=0 \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0=0$$

$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0=\frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0=-\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0.$$

$$E_0=\frac{\omega}{kc}B_0, \qquad B_0=\frac{\omega}{kc}E_0,$$



a_1, a_2 and k are perpendicular

Plane electromagnetic waves

Substitution in Maxwell's equation

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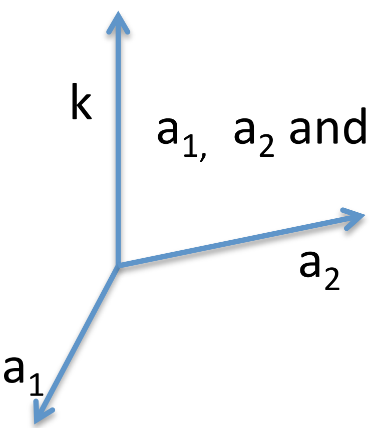
$$i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 = \frac{i\omega}{c}\hat{\mathbf{a}}_2 B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 = -\frac{i\omega}{c}\hat{\mathbf{a}}_1 E_0.$$

$$E_0 = \frac{\omega}{kc} B_0, \qquad B_0 = \frac{\omega}{kc} E_0,$$

$$E_0 = \left(\frac{\omega}{kc}\right)^2 E_0 \Rightarrow \omega = ck.$$

$$E_0 = B_0. \qquad v_{\mathrm{ph}} = c. \Rightarrow$$

EM waves travels at speed of light



a_1, a_2 and k are perpendicular

Plane electromagnetic waves
Energy flux and energy density

Time averaged pointing vector $$\langle S \rangle = \frac{c}{8\pi} \mathrm{Re}(E_0 B_0^*).$$

Since $E_0 = B_0.$

$$\langle S \rangle = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2$$

Similarly time averaged energy density

$$\langle U \rangle = \frac{1}{16\pi} \mathrm{Re}(E_0 E_0^* + B_0 B_0^*)$$

$$\langle U \rangle = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2$$

Plane electromagnetic waves
Energy flux and energy density

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Radiation spectrum

$$\Delta\omega\Delta t > 1$$

$$\downarrow$$

with a radiation field of length Δt we can define spectrum with in Δω

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt.$$

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega)e^{-i\omega t}d\omega.$$

Energy per unit area per unit time

$$\frac{dW}{dt\,dA} = \frac{c}{4\pi}E^2(t)$$

Total Energy per unit area

$$\frac{dW}{dA} = \frac{c}{4\pi}\int_{-\infty}^{\infty} E^2(t)dt$$

Radiation spectrum

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 2\pi \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega.$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} E^2(t)\,dt = 4\pi \int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega.$$

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t)\,dt \longrightarrow \frac{dW}{dA} = c \int_{0}^{\infty} |\hat{E}(\omega)|^2 d\omega,$$

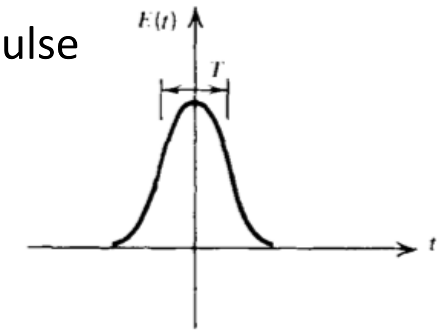
Energy per unit area per unit frequency $\longrightarrow$ $\dfrac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2$

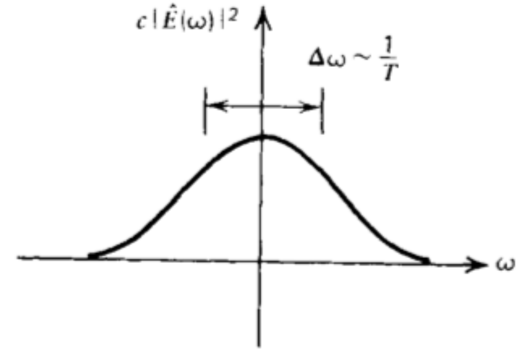
Radiation spectrum

Electric field

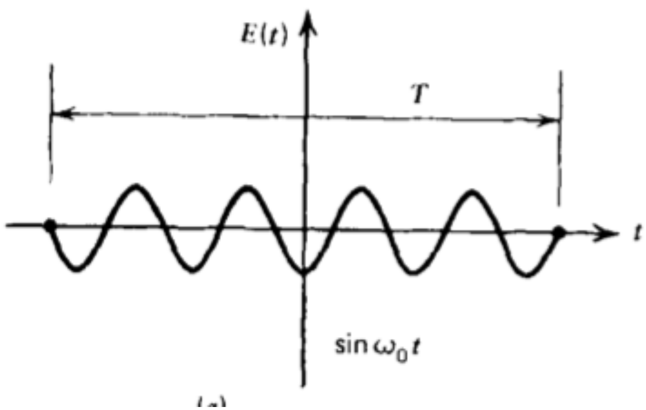
Power spectrum

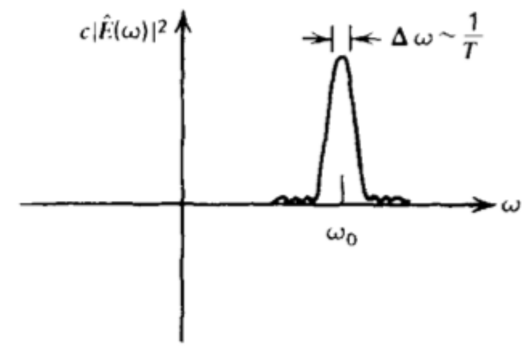
a) pulse





b) sinusoidal pulse





Time extent of pulse T determines width of finest features : $\Delta\omega \sim 1/T$

Sinusoidal time dependence in pulse shape causes spectrum concentrated near $\omega \sim \omega_0$

Observables

From an empiricist's point of view there are 4 observables for radiation

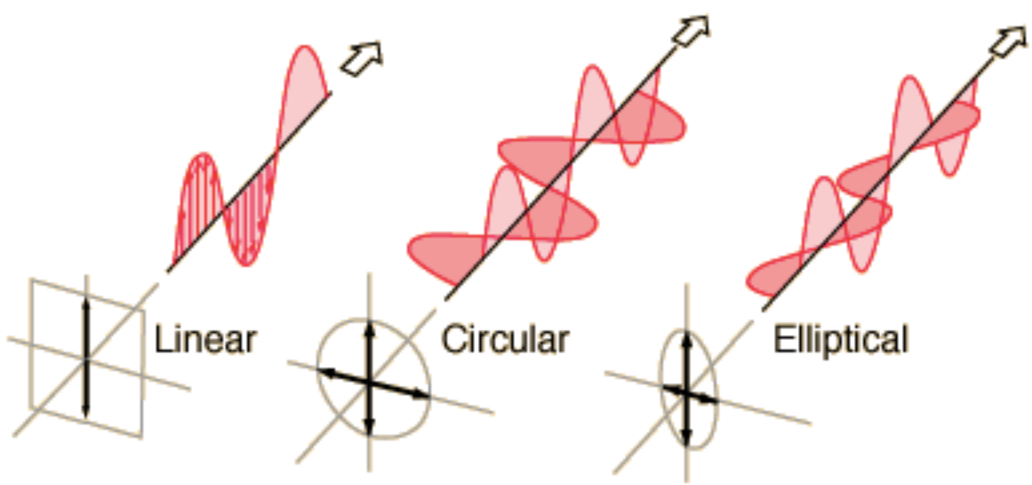
- Energy Flux
- Direction
- Frequency
- **Polarization**

Polarimetry : study of polarization of incoming radiation

Polarization of electromagnetic radiation

- ✓ Polarization is produced in various ways, including directly from some radiation processes (e.g. cyclotron and synchrotron emission), from differential absorption of radiation passing through the interstellar medium, and perhaps most commonly from the scattering of radiation.

- ✓ Property of a wave to have its Electric Field oscillating in a single plane (plane polarized wave) or in a rotating plane (elliptically or even circular polarized wave).



Polarization of electromagnetic radiation

- ✓ Polarization is produced in various ways, including directly from some radiation processes (e.g. cyclotron and synchrotron emission), from differential absorption of radiation passing through the interstellar medium, and perhaps most commonly from the scattering of radiation.

- ✓ Fractional polarizations detected from astronomical objects can be
  very high
  (pulsars: almost fully linearly polarised) to,
  very low
  (sun: one of the most sensitive polarization measurements ever made was by James Kemp in 1987, who showed that the fractional linear polarization of light from the Sun was ~ 10^{-7})

Polarization of electromagnetic radiation

- ✓ Polarimetry, is a method used to study the polarization of incoming radiation and can provide substantial clues to the nature of the source.

- ✓ Polarimetry is used to extract information such as the strength of magnetic fields in the interstellar medium (ISM), provide evidence for inflation by observations of the CMB polarization, motivate a unified model for active galactic nuclei (AGN), probing emission geometry for pulsars etc.

Polarization of electromagnetic radiation

- ✓ Study of polarization of electromagnetic plane waves from astrophysical sources and modification of the polarization in the medium.
- ✓ Plane waves are described by oscillating electric and magnetic fields, whose field vectors are orthogonal to each other and the direction of propagation.
- ✓ By convention, astronomers describe the polarization of light only in terms of the electric field vector (because E and B are orthogonal).

Maximum observed or expected degree of polarization for different astronomical objects

Radio	
galactic continuum	70%
quasars (integrated / resolved)	15% / 70%
Crab nebula	30%
pulsars (linear / circular)	80% / 70%
Optical	
planets	$> 20\%$
interstellar dust acting on starlight (linear)	10%
interstellar dust acting on starlight (circular)	0.05%
Sun and A_p stars (Zeeman effect)	100%
white dwarfs (Zeeman effect)	12%
symbiotic stars (Raman scattering)	8%
reflection nebulae (including Herbig–Haro and bipolar	60%
post–AGB stars and proto–PN (global polarisation)	30%
synchrotron (Crab nebula, blazars)	50%
synchrotron (extragalactic jets)	20%
Crab pulsar	10%
X–ray (mainly 'expected')	
solar flares	5%
Crab nebula	15%
accreting X–ray pulsars	80%
rotation–powered X–ray pulsars	10%
black hole (Lense–Thirring effect Cyg X–1)	2%
active galactic nuclei	20%
Seyfert accretion disc reprocessing	5%
γ–ray ('expected')	
pulsars	100%

Credit: Agnieszka Słowikowska

These are approximate numbers

May not be updated

Stokes parameters

- ✓ **The polarization can be described by the shape that the tip of E traced out over the course of a period, and it can be linear, circular, or elliptical.**

Stokes parameters were defined by George Gabriel Stokes in 1852, as a mathematically convenient alternative to the more common description of incoherent or partially polarized radiation in terms of its total intensity (I), (fractional) degree of polarization (p), and the shape parameters of the polarization ellipse

Polarization of electromagnetic radiation

Specific case

We discussed about monochromatic plane wave

$$\mathbf{E} = \hat{\mathbf{a}}_1 E_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

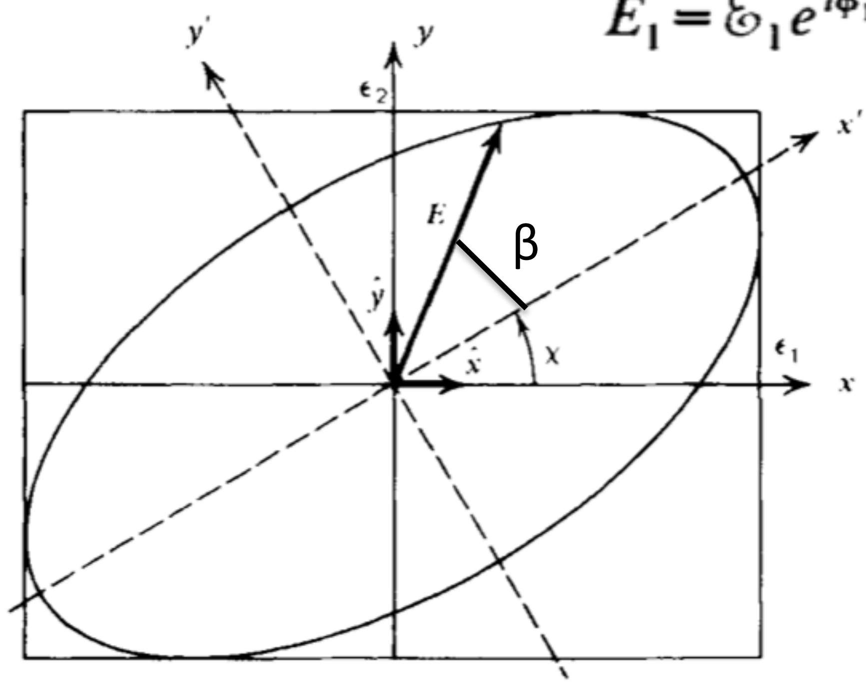
Oscillates along a_1

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E} = (\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2) e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}.$$

E_1 and E_2 are complex amplitude and can be written as

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2}$$



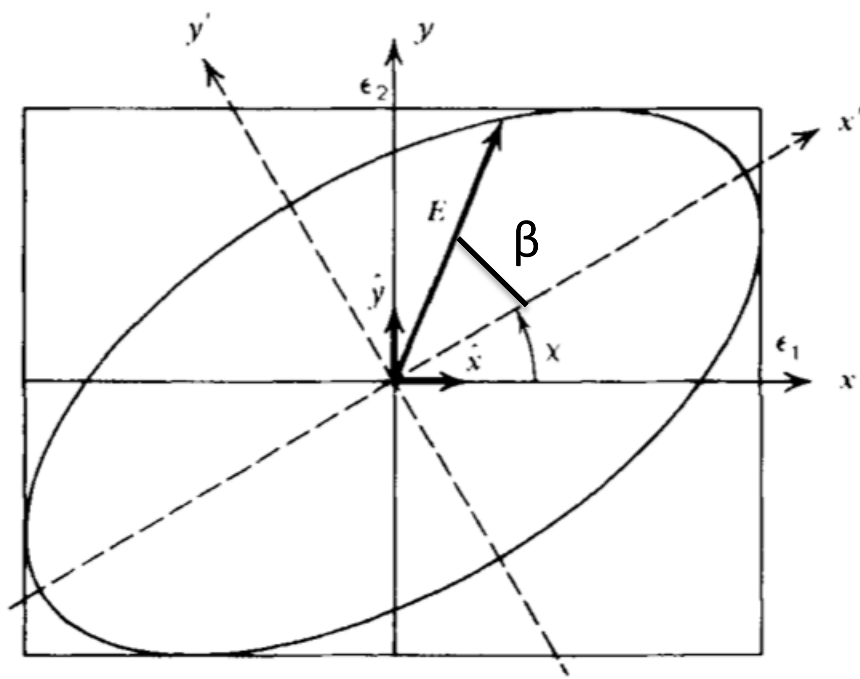
Polarization of electromagnetic radiation

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E} = (\hat{\mathbf{x}} E_1 + \hat{\mathbf{y}} E_2) e^{-i\omega t} \equiv \mathbf{E}_0 e^{-i\omega t}.$$

Considering real part of E, physical component of electric fields along x and y direction

$$\boxed{E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2).}$$



These equations describe Tip of E in x-y plane

Polarization of electromagnetic radiation

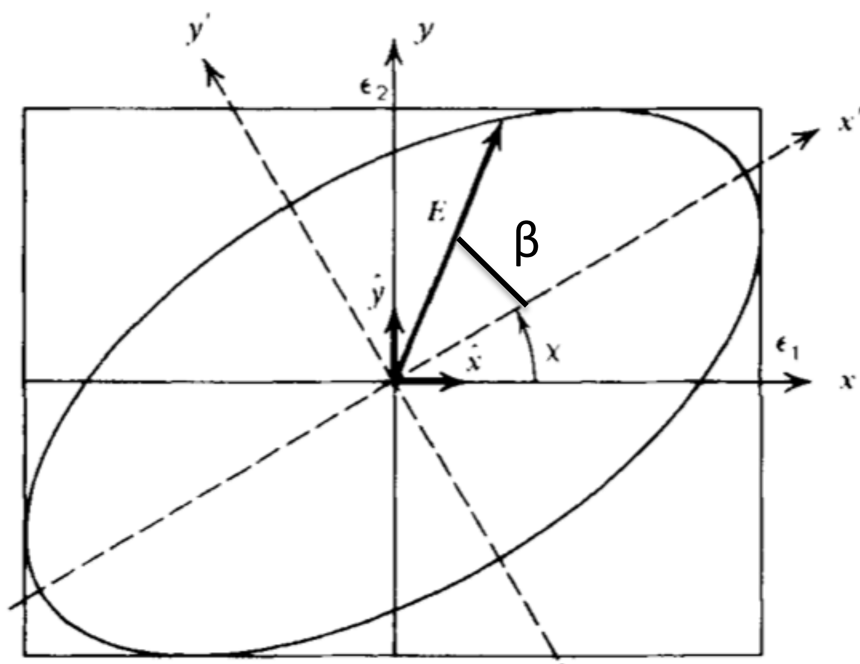
Equations describing tip of **E** in x-y plane

$$\boxed{E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2).}$$

Figure traced out by tip of **E** is an ellipse

Equations for a general ellipse relative to its principal axes x' and y'

$$E_x' = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E_y' = -\mathcal{E}_0 \sin\beta \sin\omega t$$



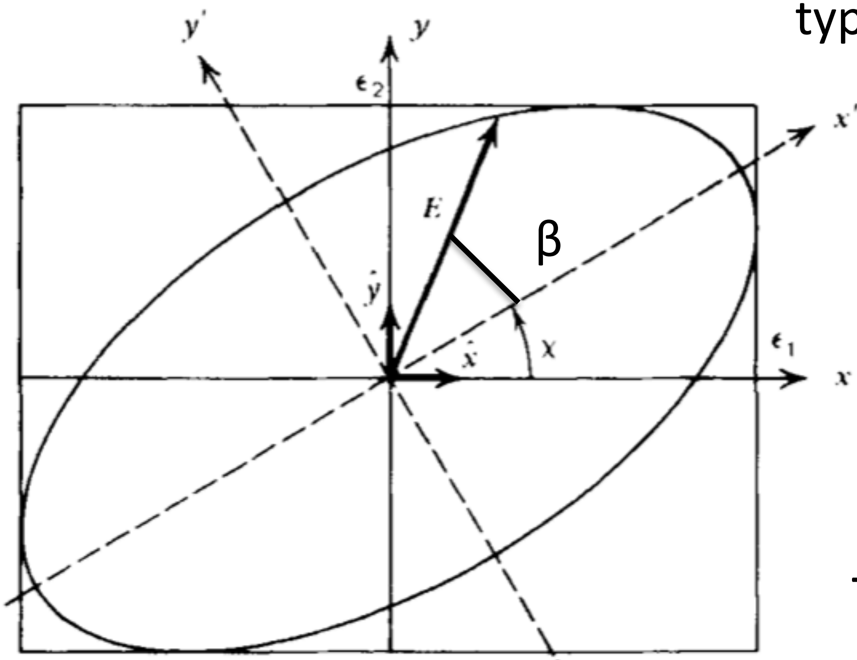
Polarization of electromagnetic radiation

Superposition of two such oscillations in perpendicular direction

$$\mathbf{E}=(\hat{\mathbf{x}}E_1+\hat{\mathbf{y}}E_2)e^{-i\omega t}\equiv\mathbf{E}_0e^{-i\omega t}.$$

Elliptically Polarized

$$E_x'=\mathcal{E}_0\cos\beta\cos\omega t,\qquad E_y'=-\mathcal{E}_0\sin\beta\sin\omega t$$



Equation of tip of electric field vector determines type of polarisation

$$(E_x'/\mathcal{E}_0\cos\beta)^2+(E_y'/\mathcal{E}_0\sin\beta)^2=1$$

↓

Elliptically Polarized

$0<\beta<\pi/2$ ⟶ Clockwise ellipse
Right-handed polarization

$-\pi/2<\beta<0$ ⟶ Anti-Clockwise ellipse
Left-handed polarization

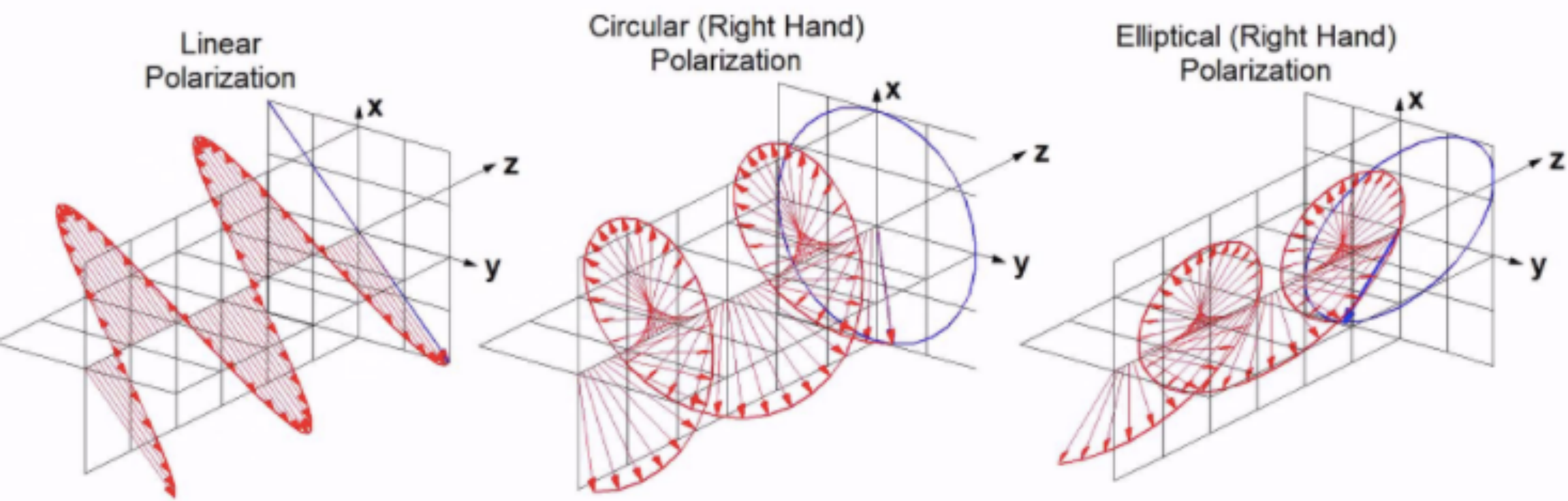
Polarization of electromagnetic radiation

Two special cases of elliptical polarization

$\beta = \pm\pi/4$ → Circular

$\beta = 0$, $\beta = \pi/2$ → Linear

Right-handed circularly polarized
Left handed circularly polarized



Polarization and stokes parameters

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$

Thus

$$E_x = \mathcal{E}_0(\cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t)$$
$$E_y = \mathcal{E}_0(\cos\beta\sin\chi\cos\omega t - \sin\beta\cos\chi\sin\omega t)$$

Polarization and stokes parameters

$$E'_x = \mathcal{E}_0 \cos\beta \cos\omega t, \qquad E'_y = -\mathcal{E}_0 \sin\beta \sin\omega t$$

Thus

$$E_x = \mathcal{E}_0(\cos\beta \cos\chi \cos\omega t + \sin\beta \sin\chi \sin\omega t)$$

$$E_y = \mathcal{E}_0(\cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t)$$

However (in slide 34),

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \qquad E_2 = \mathcal{E}_2 e^{i\phi_2}$$

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \qquad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2)$$

Consider,

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi,$$

$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi,$$

$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi,$$

$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi.$$

Polarization and stokes parameters

$$\mathcal{E}_1 \cos\phi_1 = \mathcal{E}_0 \cos\beta \cos\chi,$$

$$\mathcal{E}_1 \sin\phi_1 = \mathcal{E}_0 \sin\beta \sin\chi,$$

$$\mathcal{E}_2 \cos\phi_2 = \mathcal{E}_0 \cos\beta \sin\chi,$$

$$\mathcal{E}_2 \sin\phi_2 = -\mathcal{E}_0 \sin\beta \cos\chi.$$

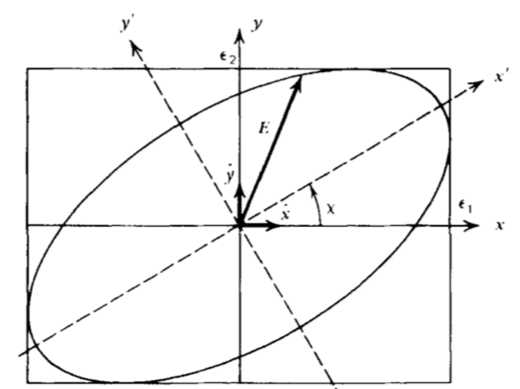
Stokes parameters

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2$$

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi$$

$$U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$

$$V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$



Polarization and stokes parameters

Stokes parameters

$$\boxed{I^2 = Q^2 + U^2 + V^2}$$

Valid for
Monochromatic wave

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2$$

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi$$

$$U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi$$

$$V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$

$$\mathcal{E}_0 = \sqrt{I}$$

$$\sin 2\beta = \frac{V}{I}$$

$$\tan 2\chi = \frac{U}{Q}.$$

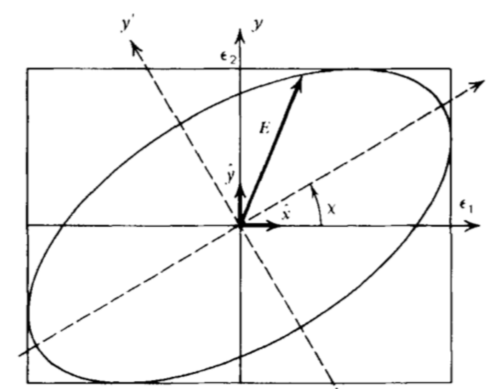
- ✓ I is Proportional to intensity of wave (+ve)

- ✓ Circularity parameter measure ratios of axes of the ellipse

+ve for Right-handed polarization
-ve for left handed polarization
V=0 for linear polarization

- ✓ Q / U measures orientation of ellipse relative to x-axis

Q=U=0 for circular polarization



Polarization and stokes parameters

Quasi monochromatic waves, $E_1(t) = \mathcal{E}_1(t)e^{i\phi_1(t)}, \qquad E_2(t) = \mathcal{E}_2(t)e^{i\phi_2(t)}$

$$I \equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle$$

$$Q \equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \rangle$$

$$U \equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = \langle 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle$$

$$V \equiv \frac{1}{i}(\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle) = \langle 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \rangle$$

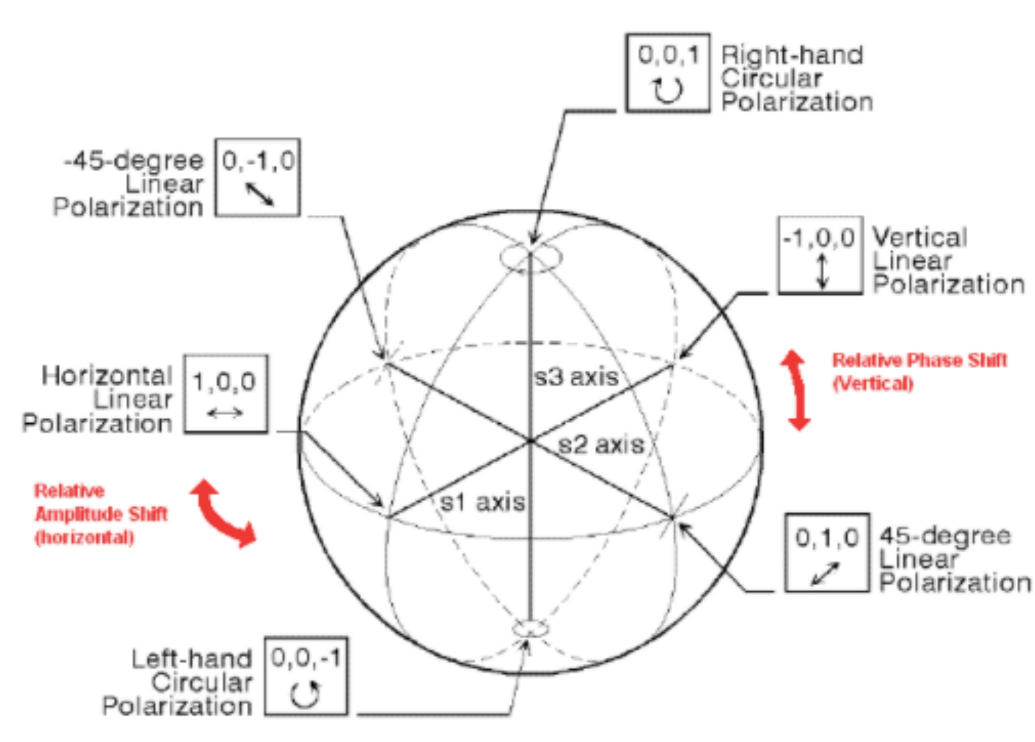
$$I^2 \geqslant Q^2 + U^2 + V^2$$

Degree of polarization,

$$\Pi \equiv \frac{I_{\mathrm{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

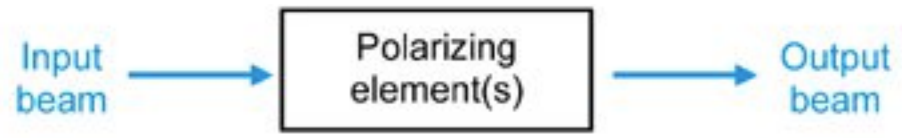
Further reading

Poincare Sphere : a graphical tool to visualize different types of polarized radiation



Further reading

Mueller Matrix : Method for transforming Stokes parameters



$$\begin{pmatrix} S_0' \\ S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

End of Lecture 4

Next lecture : 21th August