Bull. Astr. Soc. India (2012) **40**, 41–50

Kippenhahn and Thomas averaging method for the structure of rotating stars

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Received 2011 October 28; accepted 2012 January 30

Abstract. In 1970, Kippenhahn and Thomas proposed an averaging technique to study the equilibrium structures of the rotationally and tidally distorted stars. This technique has been subsequently used in the literature by various authors to study the equilibrium structures, evolutionary tracks and oscillations of rotating stars and stars that are rotationally and tidally distorted (that is, stars in binary systems). In this review paper, we have tried to see how this technique has been used by different authors in studying the various problems of rotating stars and stars in binary systems and also, how it has been modified over many years in the face of certain challenges and problems arising in the field of stellar structure and evolution.

Keywords : methods: analytical – methods: numerical – binaries: close – stars: rotation

1. Introduction

A detailed understanding of the formation, structure and evolution of stars, including the effects of rotation and magnetic field continues to be one of the main problems of the modern astrophysics. In this review we focus on one of the aspects of the structure of rotating stars. Attempts in computing rotating stellar models is as old as our interest in stellar evolution itself. In the past,

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the role of rotation in the study of stellar structure and evolution has been neglected as it leads to substantial increase in the numerical complexity of the stellar structure equations and also, because the non-rotating models have been successful in explaining the relevant observational data. However, failure of the stellar models to match the observations in detail has been partly responsible for an increased interest in effects that have been traditionally thought of as secondary, such as rotation.

The general effect of rotation has been understood for a long time (Milne 1923; Von Zeipel 1924; Chandrasekhar 1939). Rotation directly affects the structure and evolution of stars in two ways. First, it causes departure from spherical symmetry through the addition of the centrifugal forces and secondly, rotation can also give rise to the secular and dynamical instabilities that act to redistribute the angular momentum throughout the star. The redistribution of angular momentum will affect the structure via rotation distortion and the associated mass motion will cause chemical mixing that can have profound effect on the evolution.

A number of procedures for computing the rotating stellar models have been developed over the years, each with its advantages and disadvantages. The first method to study the effects of rotation and/or tidal distortions was the first order perturbation technique given by Chandrasekhar (1933a,b,c). This type of analysis was later applied by Sweet & Roy (1953) to a Cowling model star. The first order perturbation technique of Chandrasekhar was extended to the second order by Anand (1968) and Geroyannis, Tokis & Valvi (1979), and to the third order by Geroyannis & Valvi (1985).

Although a large number of authors have studied the problems of the effect of rotation and/or tidal distortions on the structure and evolution of stars, we can still classify the major techniques that have emerged during the last few decades. However, in this review, we are interested in the methods that have been used by different authors in the last few decades, in particular the Kippenhahn & Thomas (1970) averaging technique.

2. Approximation methods

The approximation methods that have been used widely by different authors in the last few decades to study the equilibrium structures, evolution and other problems of rotating stars and stars in the binary systems are disussed here.

2.1 Double approximation method

This method was developed by Monaghan & Roxburgh (1965), and does not have the obvious limitations of the perturbation method. In this method, a star is divided into two parts (i) a core which is assumed to be rotating slowly such that a perturbation technique can be applied there, (ii) a highly distorted envelope which contains a negligible amount of mass such that a Roche potential may be used here. It has been used by a number of investigators such as Roxburgh, Griffith & Sweet (1965), Faulkner, Roxburgh & Strittmatter (1968), Sackmann & Anand (1970), Martin (1970), Naylor & Anand (1972), each of whom has introduced minor modifications in the original method. This method is valid for the stars having uniform rotation; however, it is not applicable for the differentially rotating stars with rapid rotating cores.

2.2 Self Consistent Field (SCF) method

This method was devised by Ostriker & Mark (1968) and subsequently it was developed by Mark (1968) and Ostriker & Bodenheimer (1968). This method is primarily used for obtaining the accurate solutions of the total potential and hydrostatic equilibrium equation. This method iterates between solution for the density distribution and total potential. There have been modifications to the original method by Jackson (1970), Clement (1974), Clement (1978) and Hachisu (1986a,b), but method suffers from the problem that there are occasions (for very flattened structures or configurations possessing a high density contrast, Eriguchi & Muller (1985)) for which SCF method does not converge. Also, as this method solves the potential equation more accurately, it is time consuming and complex computationally. However, it remains a very powerful method for testing the more approximate techniques.

2.3 Kippenhahn and Thomas (KT) averaging method

To study the effects of rotation and tidal distortions on the equilibrium structure of gaseous spheres, Kippenhahn & Thomas (1970) developed the concept of the topologically equivalent spherical surfaces corresponding to the actual equipotential surfaces of a rotationally and tidally distorted model. This method incorporates the effects of rotation and/or tidal distortions into the basic equations of the stellar structure. In this method, the total potential (gravitational and centrifugal) and zenith distance are taken as two independent variables, so that most of the physical quantities such as density are functions of the total potential only. Assuming a potential field, other quantities which also depend on the zenith distance are averaged over equipotential surfaces. In this way, the problem is reduced to a problem in 1-dimensional space.

There is another method called the J^2 method. The detailed description of this method has been given in Papaloizou & Whelan (1973). It has been used by Whelan, Papaloizou & Smith (1971) and Whelan (1972). The stellar structure equations used in this method are formally equivalent to those as that used in the KT method. The only difference arises in the evaluation of the total potential. In this method, the Roche approximation has been used. However, the Roche approximation breaks down if the ratio of the mean interior density to central density and the ratio of the angular velocity to critical angular velocity are both significant in comparison to unity.

3. Reason for choosing KT method

The reason that we have chosen KT method for our current study is mainly due to its one dimensional approach and mathematical simplicity. This method modifies the traditional stellar structure equations with the terms that depend on the departure from the spherical symmetry. Also, the Roche potential is not used unlike in the Double Approximation method. Once the modified 1-dimensional stellar structure equations have been solved, the departure terms and equipotential surfaces can be more accurately determined and the entire process is iterated until the satisfactory convergence is obtained unlike in the SCF method.

The 1-dimensional nature of this method allows the use of traditional stellar structure and evolution codes to calculate the rotating and/or tidally distorted models in various stages of evolution and also, this modified problem in 1-dimension is easy to solve, both mathematically and computationally.

4. Use of KT method in literature

The method proposed by Kippenhahn and Thomas has been widely used in the literature to study the various problems of the rotating stars and stars in the binary systems. Bierman & Thomas (1972) has used the program described by the KT (for the computation of structure of non spherical spheres) in a method that construct the zero age models for the contact binaries and also, allow the different adiabatic constants in the convective envelope. Smith (1973) presented an extension of the work of the KT where a uniformly continuous geometry is defined in terms of the appropriate spherical model with the Roche characteristics at the surface of the configuration and sphericity at center. However, his improvement reflects departure from the simplicity of formulation by the KT. In a research note, Aranda & Thomas (1975) have clarified the inconsistency - in the results of the different authors - for the stellar model computations of the chemically homogeneous stars of one solar mass in uniform rotation, that do not agree with the results of KT (decrease in the polar radii near break up velocity).

Celikel & Ezer (1989) have used the KT method to study the effect of rotation on the evolutionary behaviors of $2.31M_{\odot}$ stars through the pre main sequence evolution as well as the zero age main sequence. They also extended the evolutionary studies up to core hydrogen exhaustion in order to obtain a theoretical model corresponding to the given mass and radius of the rotating primary component of the binary YZ Cassiopeiae. Fujimoto (1993) used the KT method while studying the evolution of accreting stars with turbulent mixing; whereas, Dominguez, Straniero & Tornambe (1996) has used this method while studying the effect of the stellar rotation on the late evolution of the intermediate mass stars. Basu, Pinsonneault & Bahcall (2000) has also used the KT method to treat the structural effects of the rotation while investigating the systematic uncertainties in determining the profiles of the solar sound speed, density and adiabatic index using the helioseismological techniques. In a series of papers, Huang (2004a,b) used the KT method to study the evolution of the rotating binary stars consisting of 9 and $6M_{\odot}$ stars in the mass transfer phase and evolution of $8M_{\odot}$ stars with rotation, taking into account the rotational mixing, respectively. Petrovic, Langer & Van der Hucht (2005) used the KT approach to implement the influence of the centrifugal force in the rotating model while studying the current massive star evolutionary models of the solar metallicity to determine which massive star physics is capable of producing the essential ingredients for gamma ray bursts for the collapsar model.

However, some authors have used the KT formalism in some modified manner, and have subsequently used this method in their further studies, or their modified formalisms have been used widely by other authors in the literature. We have categorized such studies in four sections, which are discussed here.

4.1 Endal and Sofia formalism

In a sequence of papers, Endal & Sofia (1976, 1978) made certain modifications in the KT method. Their method uses the mass, M_ψ, contained within the equipotential surface as an independent variable. The angular momentum distribution is used to solve the shape of equipotential surface, and the distortion terms are calculated that are added to the standard stellar structure equations. The modified equations are then solved in a normal way. This modified formalism of the KT method has been widely used in the literature by various authors. Law (1981) has used this modified formalism to study the effects of rotation on the internal structure of axisymmetric, differentially rotating main sequence stars. Mendes, Antona & Mazzitelli (1999) have used this modified formalism in the ATON stellar structure code so as to compute the effects of rotation on the lithium depletion of the low mass, pre main sequence stars. Landin et al. (2006) have used it to implement the rotation in ATON code while computing the rotational evolution in the pre main sequence evolutionary tracks including rotation, non gray boundary conditions and either low or high convection efficiency. Landin, Mendes & Vaz (2009) have used this modified method along with the Claircut - Legendre expansion for the gravitational potential of a self gravitating body to study the combined effects of tidal and rotational distortions on the equilibrium configuration of the low-mass pre main sequence stars.

Pinsonneault et al. (1989) have also used the modified approach followed by Endal & Sofia (1976) in the initial conditions and in the transport of the angular momentum equations of a evolution code of the rotating stellar models (known as Yale Rotating Stellar Evolution code, YREC). This modified formalism to study the structural effects of rotation in YREC code has been used by Pinsonneault, Kawaler & Demarque (1990), Chaboyer, Demarque & Pinsonneault (1995), Barnes & Sofia (1996), Krishnamurthi et al. (1997), Heger, Langer & Woosley (2000), Sills, Pinsonneault & Terndrup (2000), Barnes, Sofia & Pinsonneault (2001), Brott et al. (2011) and Yoon, Dierks & Langer (2012) in the studies of the stellar structure and evolution, each with certain modifications in the original code.

In their formalism, Endal & Sofia (1976, 1978) have included the effects of rotation by calculating the structure on the equipotential surfaces and using the diffusion approximation to take into account the effect of circulation on redistributing the angular momentum and mixing of the chemical composition. They assumed that the rotation rate is constant over the equipotential surface. However, the primary difficulty with their approach is that the angular momentum redistribution is performed by enforcing marginal stability conditions for the dynamical processes and by generating diffusion equations for the secular processes. Neither of these approaches may be correct in detail.

4.2 Mohan and Saxena formalism

Mohan & Saxena (1983, 1985) devised a methodology (hereafter, we will refer it as MS methodology) that uses the KT averaging method along with the Kopal's method of evaluating the various parameters on the Roche equipotential to determine the effects of rotation and tidal distortions on the shapes, structures and periods of small adiabatic radial and non radial modes of the oscillations of the stellar models of stars. This MS methodology has been used subsequently in the literature. Mohan & Agarwal (1987) have used this methodology to compute the effects of rotation and tidal distortions on the periods of small adiabatic radial and non radial modes of the oscillations of a series of the composite models of the stars. Mohan, Saxena & Agarwal (1990, 1991) also used the MS methodology to determine the equilibrium structures and eigenfrequencies of certain barotropic modes of certain rotationally and tidally distorted models of 10, 5 and $2.5M_{\odot}$ main sequence stars. Later, Mohan, Lal & Singh (1992, 1994, 1998) and Lal, Pathania & Mohan (2009b) have used the MS methodology to study the equilibrium structures and periods of oscillations of differentially rotating polytropic models of the stars. Lal, Mohan & Singh (2006) have used the MS methodology to obtain the structures and some observable parameters of certain differentially rotating and tidally distorted binary systems whose primary component is assumed to be a white dwarf star. Lal, Pathania & Mohan (2009a) and Pathania, Lal & Mohan (2012) have also used this methodology to study the effects of the Coriolis force on the shapes and oscillations of the rotating stars and stars in binary systems. In all the above studies, authors have used the Mohan and Saxena stellar code to determine the equilibrium structures and periods of oscillations of rotationally and/or tidally distorted stars.

4.3 Claret and Gimenez formalism

Claret & Gimenez (1993) have used the formalism of KT to evaluate the effects of rotation on the internal structure constants to compare the most modern theoretical models of the stellar structure including moderate core overshooting and mass loss and the best available data of apsidal motion rates for the double lined eclipsing binaries. Martin & Claret (1996) have used this method to investigate the influence of the stellar rotation on the Lithium depletion in low-mass pre main sequence stars. Claret (1999) has used the KT method, with some modifications to their earlier work Claret & Gimenez (1993) to take into account the effects of rotation on the internal structure of the star, in particular the theoretical apsidal motion rate. For this purpose, they have used the models with masses 2, 7 and $15M_{\odot}$ since they are representative of the mass range where the apsidal motion is detected in the binary stars. Claret & Willems (2003) have used this method - with some numerical modifications to take into account the contribution of the rotational distortion to the total potential of the star - to calculate the rotating models assuming that star has solid body rotation. They have introduced the rotating stellar models (homogeneous and evolved ones) to investigate the effects of rotationally induced changes - in the internal structure of star - on the apsidal motion in a close binary system, within the framework of the theory of the dynamic tides. Claret (2006) has used the KT formalism to analyze the evolutionary status of the double-lined eclipsing binary EK Cep. In their analysis, by using the effective temperature ratio, masses, radii, apsidal-motion rate and lithium depletion, they have found that the standard models are able to fit the physical properties of EK Cep simultaneously at the same isochrone. In the case of the rotating models they have also found acceptable solutions without the need for a very fast rotating core. Claret & Gimenez (2010) have updated their apsidal motion test of the stellar structures and evolution. For most of their studies related to the stellar structures and evolution, the authors have used Granada stellar code.

4.4 Meynet and Maeder formalism

The KT method is an excellent method to treat the hydrostatic effects of rotation in a 1-D stellar structure and evolution codes. In this method, it is assumed that the rotation velocity is spherically symmetric. However, according to Meynet & Maeder (1997), this method applies only in the cases when the angular velocity distribution has spherical symmetry (or the conservative potential exists), because in such conditions, as shown by Zahn (1992), a law $\Omega = \Omega(r)$, where omega is a function of only the distance r to the stellar center, is likely to occur. Such a rotation law called 'shellular rotation' results from the strong horizontal 'meteorological' turbulence that homogenizes rotation on the level surfaces. Meynet & Maeder (1997) have modified the KT system of equations for the case of shellular rotation. In this case, surfaces $\Omega(r) = constant$ are no longer equipotential surfaces but are isobaric level surfaces for which they have given a consistent and full system of the equations. Maeder (2003) has strengthened the validity of their modified approach when they derived a new expression for the diffusion by horizontal turbulence in rotating stars. This new estimate is up to two orders of magnitude larger than that given by previous expression. As a result, the differential rotation on an equipotential surface is found to be very small that reinforces Zahn's hypothesis of shellular rotation. Authors have mostly used the Geneva Stellar Evolution Code for their studies.

The modified formalism as that given by Meynet & Maeder (1997) for the case of shell like rotation has been widely used in the literature. Using this modified formalism, Meynet & Maeder (2000) have constructed the grids of models for the massive rotating stars in the range of 9 and $120M_\odot$ at the solar metallicity by including the various effects of rotation. Denissenkov & Vandenberg (2003) have used this modified formalism while studying the linear analysis of the stability of the rotating low mass stars (1.2 and $0.8M_\odot$) evolving from the ZAMS to the red giant branch tip. Yang & Bi (2006) have used this modified formalism of the KT to obtain the diffusion coefficient for the magnetic momentum transport and material transport in a rotating solar model. Kuzmanovska (2006) has used this formalism to analyze the influence of the uniform stellar rotation on the masses and radius of convective cores of the zero age stars. Vazquez et al. (2007) have used this modified method to discuss the evolutionary synthesis models for the massive stellar populations generated with the Starburst 99 code in combination with the new set of the stellar evolution models accounting for rotation. Ekstrom et al. (2008) have used this approach to determine the mechanism which accelerate the surface of the single stars during the main sequence evolution. They have studied their dependence on the metallicity and have derived the frequency of the stars with different surface velocities in clusters of various ages and metallicities. Song, Zhong & Lu (2009) have used this modified method while studying the configuration of the two components of a binary system under the influence of the rotational and tidal distortions. Potter, Tout & Eldridge (2012) have used this modified formalism along with the formalism of Endal & Sofia (1978) to compare the detailed grids of stellar evolution tracks of intermediate and high mass stars produced using several models for rotation. The authors have compared the different models for stellar rotation on a common numerical platform with the rotating stellar evolution code RoSE.

5. Summary

In view of analytical and mathematical simplicity of the KT method, its one dimensional approach and ability for easy incorporation in the traditional stellar structure and evolution codes, this method has usually found favor with the end users who want to study the equilibrium structures, pulsation and evolution of the rotating stars and stars in the binary systems. No doubt from the time of its inception it has under gone various changes to make it compatible with the requirements of the users. In spite of this, it still retains its basic elegance.

Acknowledgements

A. Pathania expresses his gratitude to the North-West University, Mafikeng for the award of a Postdoctoral Fellowship. The authors also thank the reviewer Adrian Potter for several useful comments.

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