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Radiative transfer on X-Y geometry

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Abstract. A unique and non-negative solution of the radiative transfer equation in two dimensional X-Y geometry in scattering and absorbing media is presented. This solution facilitates asymmetric boundary conditions both in geometry and direction from the two boundaries of geometrical configurations such as stellar atmospheres and similar objects. Further, it allows inhomogeneities in the physical properties that occur at any point in the medium.

Keywords : Radiative transfer in two dimensional geometry, boundary conditions.

1. Introduction

There are several methods of solving the equation of radiative transfer. However most of these solutions are good in symmetric geometries which means symmetric boundary conditions. In the astrophysical context, several good solutions are available in plane parallel and spherically symmetric approximations of the equation of radiative transfer. In other fields such as meteorology, chemical engineering, reactor physics, oceanography etc., solutions in three-dimensional geometries are available (Oi & Liou, 1982; Stephens 1988; Evans, 1993, 1998). However these solutions are based on certain numerical approximations which do not tell us the true characteristics of the diffuse radiation field which arises out of the multiple scattering and which also influences the ionisation properties and the velocity characteristics of the medium. These procedures generate numerical artifacts which cannot be explained on the basis of the existing physical properties of the medium. In addition to these problems, the boundary conditions are subject to certain restrictions of homogeneous nature of the medium both in geometry and physical properties of the medium. The plane parallel and spherically symmetric solutions that are used in the astrophysical context suffer from the symmetric boundary conditions (see Mihalas 1978; Peraiah

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2001; Nellison & Grant 1974; Mansike & Henning 1998; Men'shchikov & Henning 1997). In reality we have geometric and physical dependent asymmetric boundary conditions. In addition to these asymmetric boundary conditions we have to deal with the inhomogeneity of the physical characteristics of the medium, such as density, chemical composition, temperature, ionisation of several chemical species, velocities of gases etc. We need to develop a solution which can deal with such complications mentioned above. In this paper, we try to obtain a solution of the equation of transfer equation in $X - Y$ two dimensional geometry as a first attempt which can take care of the physical and geometrical asymmetries and inhomogeneities mentioned above. The equation of radiative transfer in $X - Y - Z$ Cartesian coordinate system is written as

$$\mu \frac{\partial u_\nu}{\partial x} + \eta \frac{\partial u_\nu}{\partial y} + \xi \frac{\partial u_\nu}{\partial Z} = j_\nu - \kappa_\nu u_\nu \tag{1}$$

where u_ν is the specific intensity of radiation with frequency ν, j_ν and κ_ν are the emission and absorption coefficients respectively. The quantities μ, η and ξ are the direction cosines of the ray with respect to the coordinate axes $X - Y - Z$, (see Peraiah 2001, Chapter 2, Page 34), and are related by the following relation

$$\mu^2 + \eta^2 + \xi^2 = 1. \tag{2}$$

We have suppressed the time-dependent term in equation (1). Before we study the three-dimensional equation, it is necessary to understand the problem in a two-dimensional geometry. The equation (1) becomes in two-dimensional $X - Y$ geometry as,

$$\mu \frac{\partial u_\nu}{\partial x} + \eta \frac{\partial u_\nu}{\partial y} = j_\nu - \kappa_\nu u_\nu. \tag{3}$$

In the next section we briefly describe how the solution to the above equation is derived.

2. Method of obtaining solution in X-Y geometry

The equation of transfer is written in $(X, Y), (X, -Y), (-X, -Y)$ and $(-X, Y)$ quadrants. Let us divide the space into windows as follows with $-1 \le \mu, \eta < 0$ and $0 < \mu, \eta \le 1$. The geometrical and directional discretisation will produce 8 windows and 16 equations of transfer. The 8 windows are defined as follows:

$$w1 = window1 : (X, Y); -1 \le \mu < 0 : 0 < \mu \le 1 \tag{4}$$

$$w2 = window2 : (X, Y); -1 \le \eta < 0 : 0 < \eta \le 1 \tag{5}$$

$$w3 = window3 : (X, -Y); -1 \le \mu < 0 : 0 < \mu \le 1 \tag{6}$$

$$w4 = window4 : (X, -Y); -1 \le \eta < 0 : 0 < \eta \le 1 \tag{7}$$

$$w5 = window5 : (-X, -Y); -1 \le \mu < 0 : 0 < \mu \le 1 \tag{8}$$

$$w6 = window6 : (-X, -Y); -1 \le \eta < 0 : 0 < \eta \le 1 \tag{9}$$

$$w7 = window7 : (-X, Y); -1 \le \mu < 0 : 0 < \mu \le 1 \tag{10}$$

$$w8 = window8 : (-X, Y); -1 \le \eta < 0 : 0 < \eta \le 1 \tag{11}$$

Correspondingly the equations of transfer in the 8 windows and half-space angles are written. We shall define u^+ and u^- to represent $u(+\mu)$ and $u(-\mu)$ respectively in the μ range. Similarly intensities are defined in the η range. We set σ, ω, B and P as the absorption coefficient, albedo for single scattering, source function and phase function for isotropic scattering respectively. The equations of transfer are written in the (X, Y) quadrant:

$$\mu \frac{\partial u^+(x,y)}{\partial x} + \eta \frac{\partial u^+(x,y)}{\partial y} + \sigma u^+(x,y) = \sigma(1-\omega)B(x,y)$$
$$+\frac{1}{2}\sigma\omega \int_{-1}^{1} p(x,y;\mu,\mu')u^+(x,y;\mu')d\mu' \tag{12}$$

for $0 < \mu \le 1$

$$\mu \frac{\partial u^-(x,y)}{\partial x} + \eta \frac{\partial u^-(x,y)}{\partial y} + \sigma u^-(x,y) = \sigma(1-\omega)B(x,y)$$
$$+\frac{1}{2}\sigma\omega \int_{-1}^{1} p(x,y;\mu,-\mu')u^-(x,y;\mu')d\mu' \tag{13}$$

for $-1 \le \mu < 0$

$$\eta \frac{\partial u^+(x,y)}{\partial x} - \mu \frac{\partial u^+(x,y)}{\partial y} + \sigma u^+(x,y) = \sigma(1-\omega)B(x,y)$$
$$+\frac{1}{2}\sigma\omega \int_{-1}^{1} p(x,y;\eta,\eta')u^+(x,y;\eta')d\eta' \tag{14}$$

for $0 < \eta \le 1$

$$-\eta \frac{\partial u^-(x,y)}{\partial x} + \mu \frac{\partial u^-(x,y)}{\partial y} + \sigma u^-(x,y) = \sigma(1-\omega)B(-x,y)$$
$$+\frac{1}{2}\sigma\omega \int_{-1}^{1} p(x,y;-\eta,\eta')u^-(x,y;\eta')d\eta' \tag{15}$$

for $-1 \le \eta < 0$. We shall write equations in the $(X, -Y)$ quadrant:

$$\mu \frac{\partial u^+(x,-y)}{\partial x} - \eta \frac{\partial u^+(x,-y)}{\partial y} + \sigma u^+(x,-y) = \sigma(1-\omega)B(x,-y)$$
$$+\frac{1}{2}\sigma\omega \int_{-1}^{1} p(x,-y;\mu,\mu')u^+(x,-y;\mu')d\mu' \tag{16}$$

for $0 < \mu \leq 1$.

$$\mu\frac{\partial u^-(x,-y)}{\partial x}+\eta\frac{\partial u^-(x,-y)}{\partial y}+\sigma u^-(x,-y)=\sigma(1-\omega)B(x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(x,-y;\mu,-\mu')u^-(x,-y;\mu')d\mu' \tag{17}$$

for $-1 \leq \mu < 0$.

$$-\eta\frac{\partial u^+(x,-y)}{\partial x}-\mu\frac{\partial u^+(x,-y)}{\partial y}+\sigma u^+(x,-y)=\sigma(1-\omega)B(x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(x,-y;\eta,\eta')u^+(x,-y;\eta')d\eta' \tag{18}$$

for $0 < \eta \leq 1$.

$$\eta\frac{\partial u^-(x,-y)}{\partial x}+\mu\frac{\partial u^-(x,-y)}{\partial y}+\sigma u^-(x,-y)=\sigma(1-\omega)B(-x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(x,-y;\eta,-\eta')u^-(x,-y;\eta')d\eta' \tag{19}$$

for $-1 \leq \eta < 0$. We shall write equations in $(-X,-Y)$ quadrant.

$$\eta\frac{\partial u^+(-x,-y)}{\partial x}-\mu\frac{\partial u^+(-x,-y)}{\partial y}+\sigma u^+(-x,-y)=\sigma(1-\omega)B(-x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(-x,-y;\mu,\mu')u^+(-x,-y;\mu')d\mu' \tag{20}$$

for $0 < \mu \leq 1$.

$$-\eta\frac{\partial u^-(-x,-y)}{\partial x}+\mu\frac{\partial u^-(-x,-y)}{\partial y}+\sigma u^-(-x,-y)=\sigma(1-\omega)B(-x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(-x,-y;\mu,-\mu')u^-(x,y;\mu')d\mu' \tag{21}$$

for $-1 \leq \mu < 0$.

$$\mu\frac{\partial u^+(-x,-y)}{\partial x}+\eta\frac{\partial u^+(-x,-y)}{\partial y}+\sigma u^+(-x,-y)=\sigma(1-\omega)B(-x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(-x,-y;\eta,\eta')u^+(-x,-y;\eta')d\eta' \tag{22}$$

for $0 < \eta \leq 1$.

$$-\mu\frac{\partial u^-(-x,-y)}{\partial x}-\eta\frac{\partial u^-(-x,-y)}{\partial y}+\sigma u^-(-x,-y)=\sigma(1-\omega)B(-x,-y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1}p(-x,-y;\eta,-\eta')u^-(-x,-y;\eta')d\eta' \tag{23}$$

for $-1 \leq \eta < 0$. We shall write equations in $(-X, Y)$ quadrant:

$$\mu\frac{\partial u^{+}(-x,y)}{\partial x} - \eta\frac{\partial u^{+}(-x,y)}{\partial y} + \sigma u^{+}(-x,y) = \sigma(1-\omega)B(-x,y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1} p(-x,y;\mu,\mu')u^{+}(-x,y;\mu')d\mu' \tag{24}$$

for $0 < \mu \leq 1$.

$$-\mu\frac{\partial u^{-}(-x,y)}{\partial x} + \eta\frac{\partial u^{-}(-x,y)}{\partial y} + \sigma u^{-}(-x,y) = \sigma(1-\omega)B(-x,y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1} p(-x,y;\mu,-\mu')u^{-}(-x,y;\mu')d\mu' \tag{25}$$

for $-1 \leq \mu < 0$.

$$-\eta\frac{\partial u^{+}(-x,y)}{\partial x} - \mu\frac{\partial u^{+}(-x,y)}{\partial y} + \sigma u^{+}(-x,y) = \sigma(1-\omega)B(-x,y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1} p(-x,y;\eta,\eta')u^{+}(-x,y;\eta')d\eta' \tag{26}$$

for $0 < \eta \leq 1$.

$$\eta\frac{\partial u^{-}(-x,y)}{\partial x} + \mu\frac{\partial u^{-}(-x,y)}{\partial y} + \sigma u^{-}(-x,y) = \sigma(1-\omega)B(-x,y)$$
$$+\frac{1}{2}\sigma\omega\int_{-1}^{1} p(-x,y;\eta,-\eta')u^{-}(-x,y;\eta')d\eta' \tag{27}$$

for $-1 \leq \eta < 0$. We have,

$$B(x,y) \geq 0 \tag{28}$$

$$\sigma \geq 0 \tag{29}$$

$$0 \leq \omega \leq 1. \tag{30}$$

One should note that the above inequalities are also angle dependent and become relevant in an expanding medium. As we deal with a static medium in the present situation we do not show their angle dependency. Equations (12-27) are discretised in space and angle. We designate m and n as the subscript indices for x and y coordinates. Let $\Delta x = x_{m+1} - x_m$ and $\Delta y = y_{n+1} - y_n$. We show how equation (12) is discretised and give the results for equations (13) to (27). Equation (12) is discretised as follows:

$$\mu\Delta y(u^{+,x}_{m+1,n} - u^{+,x}_{m,n}) + \eta\Delta x(u^{+,y}_{m,n+1} - u^{+,y}_{m,n}) + \frac{\sigma\Delta xy}{4}(u^{+,x}_{m,n} + u^{+,y}_{m,n} + u^{+,x}_{m+1,n} +$$
$$u^{+,y}_{m,n+1}) = \sigma\Delta xy(1-\omega)B_{m+\frac{1}{2},n+\frac{1}{2}} + \sigma\omega\frac{\Delta xy}{4}[p^{++}c(u^{+,x}_{m+1,n} + u^{+,x}_{m,n})$$
$$+p^{+-}c(u^{-,x}_{m+1,n} + u^{-,x}_{m,n}) + p^{++}c(u^{+,y}_{m,n+1} + u^{+,y}_{m,n}) + p^{+-}c(u^{-,y}_{m,n+1} + u^{-,y}_{m,n})] \tag{31}$$

where c is the quadrature weight and

$$\Delta xy = \Delta x \Delta y \tag{32}$$

Equation (31) is rewritten as follows:

$$\begin{aligned} & g_1 u^{+,x}_{m+1,n} + g_2 u^{+,y}_{m,n+1} + g_3 u^{-,x}_{m,n} + g_3 u^{-,y}_{m,n} = g_4 \\ & + g_5 u^{+,x}_{m,n} + g_6 u^{+,y}_{m,n} + g_7 u^{-,x}_{m+1,n} + g_7 u^{-,y}_{m,n+1} \end{aligned} \tag{33}$$

The coefficient g's are given in the Appendix. The equation (13) is similarly discretised and written as follows:

$$\begin{aligned} & g_8 u^{+,x}_{m+1,n} + g_8 u^{+,y}_{m,n+1} + g_9 u^{-,x}_{m,n} + g_{10} u^{-,y}_{m,n} = g_{11} \\ & + g_{12} u^{+,x}_{m,n} + g_{12} u^{+,y}_{m,n} + g_{13} u^{-,x}_{m+1,n} + g_{14} u^{-,y}_{m,n+1} \end{aligned} \tag{34}$$

Combining equations (33) and (34) we get,

$$\begin{aligned} & \begin{bmatrix} g_1 & g_2 \\ g_8 & g_8 \end{bmatrix} \begin{bmatrix} u^{+,x}_{m+1,n} \\ u^{+,y}_{m,n+1} \end{bmatrix} + \begin{bmatrix} g_3 & g_3 \\ g_9 & g_{10} \end{bmatrix} \begin{bmatrix} u^{-,x}_{m,n} \\ u^{-,y}_{m,n} \end{bmatrix} = \begin{bmatrix} g_4 \\ g_{11} \end{bmatrix} + \\ & \begin{bmatrix} g_5 & g_6 \\ g_{12} & g_{12} \end{bmatrix} \begin{bmatrix} u^{+,x}_{m,n} \\ u^{+,y}_{m,n} \end{bmatrix} + \begin{bmatrix} g_7 & g_7 \\ g_{13} & g_{14} \end{bmatrix} \begin{bmatrix} u^{-,x}_{m+1,n} \\ u^{-,y}_{m,n+1} \end{bmatrix}. \end{aligned} \tag{35}$$

Equations (14) to (27) are similarly written and are given below:

$$\begin{aligned} & \begin{bmatrix} g_{15} & g_{16} \\ g_{22} & g_{22} \end{bmatrix} \begin{bmatrix} u^{+,x}_{m+1,-n} \\ u^{+,-y}_{m,-n-1} \end{bmatrix} + \begin{bmatrix} g_{17} & g_{17} \\ g_{23} & g_{24} \end{bmatrix} \begin{bmatrix} u^{-,x}_{-m,n} \\ u^{-,-y}_{m,-n} \end{bmatrix} = \begin{bmatrix} g_{18} \\ g_{25} \end{bmatrix} + \\ & \begin{bmatrix} g_{19} & g_{20} \\ g_{26} & g_{26} \end{bmatrix} \begin{bmatrix} u^{+,x}_{m,-n} \\ u^{+,-y}_{m,-n} \end{bmatrix} + \begin{bmatrix} g_{21} & g_{21} \\ g_{27} & g_{28} \end{bmatrix} \begin{bmatrix} u^{-,x}_{m+1,-n} \\ u^{-,-y}_{m,-n-1} \end{bmatrix} \end{aligned} \tag{36}$$

$$\begin{aligned} & \begin{bmatrix} g_{29} & g_{30} \\ g_{36} & g_{36} \end{bmatrix} \begin{bmatrix} u^{+,x}_{m+1,-n} \\ u^{+,-y}_{m,-n-1} \end{bmatrix} + \begin{bmatrix} g_{31} & g_{31} \\ g_{37} & g_{38} \end{bmatrix} \begin{bmatrix} u^{-,x}_{m,-n} \\ u^{-,-y}_{m,-n} \end{bmatrix} = \begin{bmatrix} g_{32} \\ g_{39} \end{bmatrix} + \\ & \begin{bmatrix} g_{33} & g_{34} \\ g_{40} & g_{40} \end{bmatrix} \begin{bmatrix} u^{+,x}_{m,-n} \\ u^{+,-y}_{m,-n} \end{bmatrix} + \begin{bmatrix} g_{35} & g_{35} \\ g_{41} & g_{42} \end{bmatrix} \begin{bmatrix} u^{-,x}_{m+1,-n} \\ u^{-,-y}_{m,-n-1} \end{bmatrix} \end{aligned} \tag{37}$$

$$\begin{aligned} & \begin{bmatrix} g_{43} & g_{44} \\ g_{50} & g_{50} \end{bmatrix} \begin{bmatrix} u^{+,-x}_{-m-1,-n} \\ u^{+,-y}_{-m,-n-1} \end{bmatrix} + \begin{bmatrix} g_{45} & g_{45} \\ g_{51} & g_{52} \end{bmatrix} \begin{bmatrix} u^{-,-x}_{-m,-n} \\ u^{-,-y}_{-m,-n} \end{bmatrix} = \begin{bmatrix} g_{46} \\ g_{53} \end{bmatrix} + \\ & \begin{bmatrix} g_{47} & g_{48} \\ g_{54} & g_{54} \end{bmatrix} \begin{bmatrix} u^{+,-x}_{-m,-n} \\ u^{+,-y}_{-m,-n} \end{bmatrix} + \begin{bmatrix} g_{49} & g_{49} \\ g_{55} & g_{56} \end{bmatrix} \begin{bmatrix} u^{-,-x}_{-m-1,-n} \\ u^{-,-y}_{-m,-n-1} \end{bmatrix} \end{aligned} \tag{38}$$

$$\begin{aligned} & \begin{bmatrix} g_{57} & g_{58} \\ g_{64} & g_{64} \end{bmatrix} \begin{bmatrix} u^{+,-x}_{-m-1,-n} \\ u^{+,-y}_{-m,-n-1} \end{bmatrix} + \begin{bmatrix} g_{59} & g_{59} \\ g_{65} & g_{66} \end{bmatrix} \begin{bmatrix} u^{-,-x}_{-m,-n} \\ u^{-,-y}_{-m,-n} \end{bmatrix} = \begin{bmatrix} g_{60} \\ g_{67} \end{bmatrix} + \\ & \begin{bmatrix} g_{61} & g_{62} \\ g_{68} & g_{68} \end{bmatrix} \begin{bmatrix} u^{+,-x}_{-m,-n} \\ u^{+,-y}_{-m,-n} \end{bmatrix} + \begin{bmatrix} g_{63} & g_{63} \\ g_{69} & g_{70} \end{bmatrix} \begin{bmatrix} u^{-,-x}_{-m-1,-n} \\ u^{-,-y}_{-m,-n-1} \end{bmatrix} \end{aligned} \tag{39}$$

$$\begin{bmatrix} g_{71} & g_{72} \\ g_{78} & g_{78} \end{bmatrix}\begin{bmatrix} u^{+,-x}_{-m-1,n} \\ u^{+,y}_{-m,n+1} \end{bmatrix} + \begin{bmatrix} g_{73} & g_{73} \\ g_{79} & g_{80} \end{bmatrix}\begin{bmatrix} u^{-,-x}_{-m,n} \\ u^{-,y}_{-m,n} \end{bmatrix} = \begin{bmatrix} g_{74} \\ g_{81} \end{bmatrix} + \begin{bmatrix} g_{75} & g_{76} \\ g_{82} & g_{82} \end{bmatrix}\begin{bmatrix} u^{+,-x}_{-m,n} \\ u^{+,y}_{-m,n} \end{bmatrix} + \begin{bmatrix} g_{77} & g_{77} \\ g_{83} & g_{84} \end{bmatrix}\begin{bmatrix} u^{-,-x}_{-m-1,n} \\ u^{-,y}_{-m,n+1} \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} g_{85} & g_{86} \\ g_{92} & g_{92} \end{bmatrix}\begin{bmatrix} u^{+,-x}_{-m-1,n} \\ u^{+,y}_{-m,n+1} \end{bmatrix} + \begin{bmatrix} g_{87} & g_{87} \\ g_{93} & g_{94} \end{bmatrix}\begin{bmatrix} u^{-,-x}_{-m,n} \\ u^{-,y}_{-m,n} \end{bmatrix} = \begin{bmatrix} g_{88} \\ g_{95} \end{bmatrix} + \begin{bmatrix} g_{89} & g_{90} \\ g_{96} & g_{96} \end{bmatrix}\begin{bmatrix} u^{+,-x}_{-m,n} \\ u^{+,y}_{-m,n} \end{bmatrix} + \begin{bmatrix} g_{91} & g_{91} \\ g_{97} & g_{98} \end{bmatrix}\begin{bmatrix} u^{-,-x}_{-m-1,n} \\ u^{-,y}_{-m,n+1} \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} g_{99} & g_{100} \\ g_{106} & g_{106} \end{bmatrix}\begin{bmatrix} u^{+,x}_{m+1,n} \\ u^{+,y}_{m,n+1} \end{bmatrix} + \begin{bmatrix} g_{101} & g_{101} \\ g_{107} & g_{108} \end{bmatrix}\begin{bmatrix} u^{-,x}_{m,n} \\ u^{-,y}_{m,n} \end{bmatrix} = \begin{bmatrix} g_{102} \\ g_{109} \end{bmatrix} + \begin{bmatrix} g_{103} & g_{104} \\ g_{110} & g_{110} \end{bmatrix}\begin{bmatrix} u^{+,x}_{m,n} \\ u^{+,y}_{m,n} \end{bmatrix} + \begin{bmatrix} g_{105} & g_{105} \\ g_{111} & g_{112} \end{bmatrix}\begin{bmatrix} u^{-,x}_{m+1,n} \\ u^{-,y}_{m,n+1} \end{bmatrix} \tag{42}$$

where ± signs refer to $\pm\mu$ or $\pm\eta$ and x, y refer to the X and Y axes. Further, $\pm m$ and $\pm n$ refer to $\pm x_m$ or $\pm x_n$ respectively. The emergent and incident intensities are written following equations (35) to (42) in all the 4 quadrants. These are written as follows:

$$(u^{+,x}_{m,n}, u^{+,x}_{m,-n}, u^{+,-y}_{m,-n}, u^{+,-y}_{-m,-n}, u^{+,-x}_{-m,-n}, u^{+,-x}_{-m,n}, u^{+,y}_{-m,n}, u^{+,y}_{m,n}) = \mathbf{U}^+_1(say) \tag{43}$$

and

$$(u^{-,x}_{m+1,n}, u^{-,x}_{m+1,-n}, u^{-,-y}_{m,-n-1}, u^{-,-y}_{-m,-n-1}, u^{-,-x}_{-m-1,-n}, u^{-,-x}_{-m-1,n}, u^{-,y}_{-m,n+1}, u^{-,y}_{m,n+1}) = \mathbf{U}^-_2(say). \tag{44}$$

The incident intensities are given as,

$$(u^{-,x}_{m,n}, u^{-,x}_{m,-n}, u^{-,-y}_{m,-n}, u^{-,-y}_{-m,-n}, u^{-,-x}_{-m,-n}, u^{-,-x}_{-m,n}, u^{-,y}_{-m,n}, u^{-,y}_{m,n}) = \mathbf{U}^-_1(say) \tag{45}$$

and

$$(u^{+,x}_{m+1,n}, u^{+,x}_{m+1,-n}, u^{+,-y}_{m,-n-1}, u^{+,-y}_{-m,-n-1}, u^{+,-x}_{-m-1,-n}, u^{+,-x}_{-m-1,n}, u^{+,y}_{-m,n+1}, u^{+,y}_{m,n+1}) = \mathbf{U}^+_2(say) \tag{46}$$

Equations (35) to (42) together with equations (43) to (46) can be written as,

$$\mathbf{K}\begin{bmatrix} \mathbf{U}^+_1 \\ \mathbf{U}^-_2 \end{bmatrix} = \begin{bmatrix} \mathbf{\Sigma}^+ \\ \mathbf{\Sigma}^- \end{bmatrix} + \mathbf{L}\begin{bmatrix} \mathbf{U}^-_1 \\ \mathbf{U}^+_2 \end{bmatrix}. \tag{47}$$

Thus we can write the emergent intensity vectors in terms of incident intensity vectors and source vectors as,

$$\begin{bmatrix} \mathbf{U}^+_1 \\ \mathbf{U}^-_2 \end{bmatrix} = \mathbf{K}^{-1}\begin{bmatrix} \mathbf{\Sigma}^+ \\ \mathbf{\Sigma}^- \end{bmatrix} + \mathbf{K}^{-1}\mathbf{L}\begin{bmatrix} \mathbf{U}^-_1 \\ \mathbf{U}^+_2 \end{bmatrix}. \tag{48}$$

The source vectors are

$$\mathbf{\Sigma}^{+} = (g_{109}, g_{102}, g_4, g_{11}, g_{98}, g_{25}, g_{32}, g_{39}) \tag{49}$$

$$\mathbf{\Sigma}^{-} = (g_{46}, g_{53}, g_{60}, g_{67}, g_{74}, g_{81}, g_{88}, g_{95}). \tag{50}$$

The $\mathbf{K}$ and $\mathbf{L}$ matrices are given below. We shall set,

$$\rho = \frac{\Delta x \Delta y \sigma \omega}{4} \tag{51}$$

$$\rho_1 = \rho p^{++} c, \rho_2 = \rho p^{+-} c, \rho_3 = \rho p^{-+} c, \rho_4 = \rho p^{--} c. \tag{52}$$

In the case of isotropic scattering we have,

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 \tag{53}$$

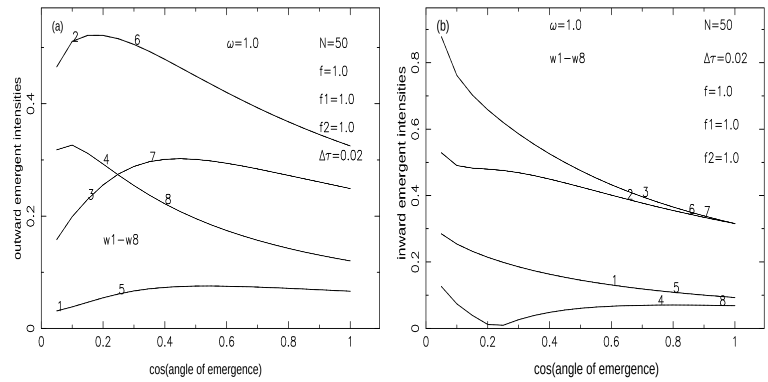

Figure 1. Outward (left) and inward (right) intensities.

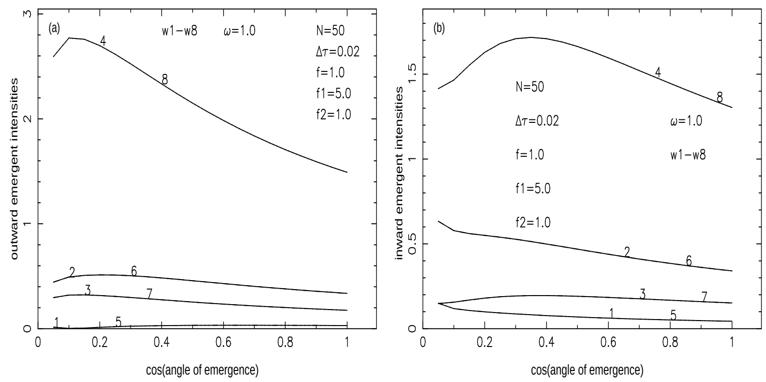

Figure 2. Outward (left) and inward (right) intensities.

The matrix **K** is given below:

$$
\begin{bmatrix}
g_5+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & g_6+\rho_1 & g_7+\rho_2 \\
 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho 2 & g_7+\rho_2 & \\
g_{12}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & g_{12}+\rho_3 & g_{13}+\rho_4 \\
 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho 4 & g_{14}+\rho_4 & \\
2\rho_1 & g_{19}+\rho_1 & g_{20}+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_2 \\
 & g_{21}+\rho_2 & g_{21}+\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho 2 & 2\rho_2 & \\
2\rho_3 & g_{26}+\rho_3 & g_{26}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_4 \\
 & g_{27}+\rho_4 & g_{28}+\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho 4 & 2\rho_4 & \\
2\rho_1 & g_{33}+\rho_1 & g_{34}+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_2 \\
 & g_{35}+\rho_2 & g_{35}+\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho 2 & 2\rho_2 & \\
2\rho_3 & g_{40}+\rho_3 & g_{40}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_4 \\
 & g_{41}+\rho_4 & g_{42}+\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho 4 & 2\rho_4 & \\
2\rho_1 & 2\rho_1 & 2\rho_1 & g_{48}+\rho_1 & g_{47}+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_2 \\
 & 2\rho_2 & 2\rho_2 & g_{49}+\rho_2 & g_{49}+\rho_2 & 2\rho_2 & 2\rho 2 & 2\rho_2 & \\
2\rho_3 & 2\rho_3 & 2\rho_3 & g_{54}+\rho_3 & g_{54}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_4 \\
 & 2\rho_4 & 2\rho_4 & g_{56}+\rho_4 & g_{55}+\rho_4 & 2\rho_4 & 2\rho 4 & 2\rho_4 & \\
2\rho_1 & 2\rho_1 & 2\rho_1 & g_{62}+\rho_1 & g_{61}+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_2 \\
 & 2\rho_2 & 2\rho_2 & g_{63}+\rho_2 & g_{63}+\rho_2 & 2\rho_2 & 2\rho 2 & 2\rho_2 & \\
2\rho_3 & 2\rho_3 & 2\rho_3 & g_{68}+\rho_3 & g_{68}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_4 \\
 & 2\rho_4 & 2\rho_4 & g_{70}+\rho_4 & g_{69}+\rho_4 & 2\rho_4 & 2\rho 4 & 2\rho_4 & \\
2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & g_{75}+\rho_1 & g_{76}+\rho_1 & 2\rho_1 & 2\rho_1 \\
 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & g_{77}+\rho_2 & g_{77}+\rho 2 & 2\rho_2 & \\
2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & g_{82}+\rho_3 & g_{82}+\rho_3 & 2\rho_3 & 2\rho_4 \\
 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & g_{83}+\rho_4 & g_{84}+\rho 4 & 2\rho_4 & \\
2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & g_{89}+\rho_1 & g_{90}+\rho_1 & 2\rho_1 & 2\rho_1 \\
 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & g_{91}+\rho_2 & g_{91}+\rho 2 & 2\rho_2 & \\
2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & g_{96}+\rho_3 & g_{96}+\rho_3 & 2\rho_3 & 2\rho_4 \\
 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & g_{97}+\rho_4 & g_{96}+\rho 4 & 2\rho_4 & \\
g_{103}+\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & 2\rho_1 & g_{104}+\rho_1 & g_{105}+\rho_2 \\
 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho_2 & 2\rho 2 & g_{105}+\rho_2 & \\
g_{110}+\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & 2\rho_3 & g_{110}+\rho_3 & g_{111}+\rho_4 \\
 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho_4 & 2\rho 4 & g_{112}+\rho_4 &
\end{bmatrix}
$$

The matrix **L** is given below:

$$
\begin{bmatrix}
g_3-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & g_3-\rho_2 & g_1-\rho_1 \\
 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & g_2-\rho_1 & \\
g_9-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & g_{10}-\rho_4 & g_8-\rho_3 \\
 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & g_8-\rho_3 & \\
-2\rho_2 & g_{17}-\rho_2 & g_{17}-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_1 \\
 & g_{15}-\rho_1 & g_{16}-\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & \\
-2\rho_4 & g_{23}-\rho_4 & g_{24}-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_3 \\
 & g_{22}-\rho_3 & g_{22}-\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & \\
-2\rho_2 & g_{31}-\rho_2 & g_{31}-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_1 \\
 & g_{29}-\rho_1 & g_{30}-\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & \\
-2\rho_4 & g_{37}-\rho_4 & g_{38}-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_3 \\
 & g_{36}-\rho_{36} & g_{22}-\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & \\
-2\rho_2 & -2\rho_2 & -2\rho_2 & g_{45}-\rho_2 & g_{45}-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_1 \\
 & -2\rho_1 & -2\rho_1 & g_{44}-\rho_1 & g_{43}-\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & \\
-2\rho_4 & -2\rho_4 & -2\rho_4 & g_{52}-\rho_4 & g_{51}-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_3 \\
 & -2\rho_3 & -2\rho_3 & g_{50}-\rho_3 & g_{50}-\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & \\
-2\rho_2 & -2\rho_2 & -2\rho_2 & g_{59}-\rho_2 & g_{59}-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_1 \\
 & -2\rho_1 & -2\rho_1 & g_{58}-\rho_1 & g_{57}-\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & \\
-2\rho_4 & -2\rho_4 & -2\rho_4 & g_{66}-\rho_4 & g_{65}-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_3 \\
 & -2\rho_3 & -2\rho_3 & g_{64}-\rho_3 & g_{64}-\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & \\
-2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & g_{73}-\rho_2 & g_{73}-\rho_2 & -2\rho_2 & -2\rho_1 \\
 & -2\rho_1 & -2\rho_1 & g-2\rho_1 & -2\rho_1 & g_{71}-\rho_1 & g_{72}-\rho_1 & -2\rho_1 & \\
-2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & g_{79}-\rho_4 & g_{80}-\rho_4 & -2\rho_4 & -2\rho_3 \\
 & -2\rho_3 & -2\rho_3 & g-2\rho_3 & -2\rho_3 & g_{78}-\rho_3 & g_{78}-\rho_3 & -2\rho_3 & \\
-2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & g_{87}-\rho_2 & g_{87}-\rho_2 & -2\rho_2 & -2\rho_1 \\
 & -2\rho_1 & -2\rho_1 & g-2\rho_1 & -2\rho_1 & g_{85}-\rho_1 & g_{86}-\rho_1 & -2\rho_1 & \\
-2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & g_{93}-\rho_4 & g_{94}-\rho_4 & -2\rho_4 & -2\rho_3 \\
 & -2\rho_3 & -2\rho_3 & g-2\rho_3 & -2\rho_3 & g_{92}-\rho_3 & g_{92}-\rho_3 & -2\rho_3 & \\
g_{101}-\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & -2\rho_2 & g_{101}-\rho_2 & g_{99}-\rho_1 \\
 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & -2\rho_1 & g_{100}-\rho_1 & \\
g_{107}-\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & -2\rho_4 & g_{108}-\rho_4 & g_{106}-\rho_3 \\
 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & -2\rho_3 & g_{106}-\rho_3 &
\end{bmatrix}
$$

We have two types of optical depths: one in the X-direction and the other in the Y-direction. These are defined as follows:

$$\Delta\tau_x = \sigma\Delta x \tag{54}$$

$$\Delta\tau_y = \sigma\Delta y. \tag{55}$$

We will further define,

$$\Delta\tau = \sigma\Delta l \tag{56}$$

where Δl is a typical length segment which can be determined from the non-negativity conditions of the reflection and transmission operators across a 'cell' (see Peraiah, 2001). We shall relate Δx, Δy and Δl by the following relation:

$$f_1\Delta x = f_2\Delta y = f\Delta l \tag{57}$$

Now $\Delta\tau_x$ and $\Delta\tau_y$ become in terms of f's and $\Delta\tau$,

$$\Delta\tau_x = \frac{f}{f_1}\Delta\tau \tag{58}$$

and

$$\Delta\tau_y = \frac{f}{f_2}\Delta\tau. \tag{59}$$

The factors f's represent the extent of inhomogeneity in the X and Y directions. When the medium is homogeneous we have,

$$f_1 = f_2 = f \tag{60}$$

If we set,

$$\alpha = \frac{f}{f_1}, \beta = \frac{f}{f_2}, \gamma = \frac{f^2}{f_1 f_2} = \alpha\beta \tag{61}$$

then,

$$\mu\Delta y = \mu\frac{f}{f_2}\Delta l = \mu\beta\Delta l \tag{62}$$

and

$$\eta\Delta x = \eta\frac{f}{f_1}\Delta l = \eta\alpha\Delta l. \tag{63}$$

Further we write,

$$s = \frac{\sigma\omega\Delta x\Delta y}{4} = \frac{\sigma\omega\gamma\Delta l^2}{4} = \frac{\gamma\omega\Delta\tau\Delta l}{4} \tag{64}$$

In the light of the above equations (57) to (66) the quantity ug_1 can be written as,

$$\begin{aligned} ug_1 &= u(\mu\Delta y + s - tp_1) = \\ &u\Delta l\left(\beta\mu + \frac{\gamma\Delta\tau}{4} - \omega pc\frac{\gamma\Delta\tau}{4}\right). \end{aligned} \tag{65}$$

Thus all intensities are multiplied by Δl. The matrices $\mathbf{K}$ and $\mathbf{L}$ are 16x16xm sized matrices, where m stands for the number of angles μ and η. To reduce the computational labour we need to use a small number of angles at specific points (such as Gauss-Legendre quadrature points on (0,1)) in the range of $0 < \mu, \eta \leq 1$. However we may need the intensities at a given point of the angles μ or η. There are two ways of obtaining the intensities at a given angle:(1) interpolation for the intermediate intensities: this process will not give accurate picture of the result and (2) calculation of the mean intensity using the existing intensities and employ the newly calculated mean intensity to estimate the intensities at any given angle by using the formal solution of the transfer equation, which would give a continuous curve for intensities vs angle of emergence. The mean intensity is calculated by the following relation,

$$J = \frac{1}{2}\int_{-1}^{+1} u(\mu)d\mu \tag{66}$$

from which we obtain the formal solution through the following relations:

$$u(0,\mu) = u(\tau,\mu)e^{-\frac{\tau}{\mu}} + \int_0^{\tau} J(t,\mu)e^{-\frac{\tau}{\mu}}\frac{dt}{\mu} \tag{67}$$

for the outward intensities and,

$$u(\tau,-\mu) = u(0,\mu)e^{-\frac{\tau}{\mu}} + \int_0^{\tau} J(t,-\mu)e^{-\frac{\tau-t}{\mu}}\frac{dt}{\mu} \tag{68}$$

for the inward intensities. This completes the solution. Equations (12) to (27) are discretised on the X-Y coordinate system. For example, equation (12) is discretised and the resultant equation is given in equation (31). Equation (13) is similarly discretised and these two discretised equations are given in equation (35). Similarly equations (14) to (27) are discretised and written in equations (36) to (42). The incident intensity vectors $\mathbf{u_1^-}$ and $\mathbf{u_2^+}$ and the emergent intensity vectors $\mathbf{u_1^+}$ and $\mathbf{u_2^-}$ are given in equations (43) to (46). The emergent intensity vectors are given in equation (48) in terms of the incident intensity vectors and the source vectors. This is in the form of the 'Interaction Principle' (see Peraiah 2001). The quantity $\mathbf{K^{-1}L}$ gives us the reflection and transmission operators in a given 'cell' which is denied as that space in which the reflection and transmission operators are non-negative and each element of these operators are less than unity and generate a continuous solution of the transfer equation. The medium is divided into several cells or shells each of which satisfy the conditions of continuity and stability of the solution. Once the operators of reflection and transmission are calculated for all the shells we can calculate the diffuse radiation field or the internal radiation field by using the scheme given in Peraiah (2001). This scheme gives us the incident and emergent intensities at the boundaries of all the shells. As we have used a small number of angles (to reduce the computational time), the number of intensities are not enough to get a good result of the solution. Therefore we use equation (68) to calculate the mean intensities which in turn are used to calculate the outward and inward intensities at the boundary of each shell through equations (69) and (70). Now, we can use any number of $\mu' s$ or $\eta' s$ so that we can obtain variation of the outward and inward directed intensities against the angles of emergence required to produce smooth variation. These are given in Figs (1) to (9).

3. Results and discussion

Equations (69) and (70) enable us to calculate the outward and inward intensities at angles in the range of $0 < \mu, \eta \leq 1$ from all the 8 windows defined in (4) to (11). We used 4 Gaussian points for the angle quadrature on $0 < \mu \leq 1$. These are: $\mu_1 = 0.06943$, $\mu_2 = 0.33001$, $\mu_3 = 0.66999$, $\mu_4 = 0.93057$, and the corresponding weights are $c_1 = 0.17393$, $c_2 = 0.32607$, $c_3 = c_2$ and $c_4 = c_1$. We have used the same roots and weights for η. The medium is divided into several shells. We assume that no radiation is incident on the outer boundary and radiation is incident from the inner boundary (similar to that of a stellar atmosphere). These are given by

$$\mathbf{U_2^+}((w1 - w8), \tau = 0) = 0. \tag{69}$$

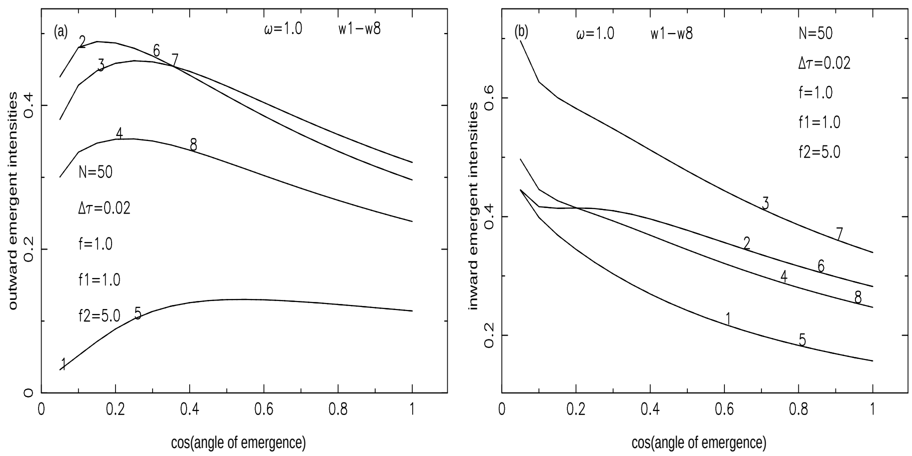

Figure 3. Outward (left) and inward (right) intensities.

This implies that no radiation is incident at $\tau = 0$ (that is from outside the atmosphere) through any of the windows w1-w8. We give the incident radiation from the inside of the medium as follows,

$$\mathbf{U_1^-}((w1 - w2), \tau = \tau_{max}) = 1. \tag{70}$$

This implies that through all the windows (and for all $\mu' s$ and $\eta' s$) we give incident radiation of unit intensity (one should note here that the incident radiation can be given individually in each window). In this case we give no internal sources that is, we set,

$$\omega = 1 \tag{71}$$

$$B^+, B^- = 0 \tag{72}$$

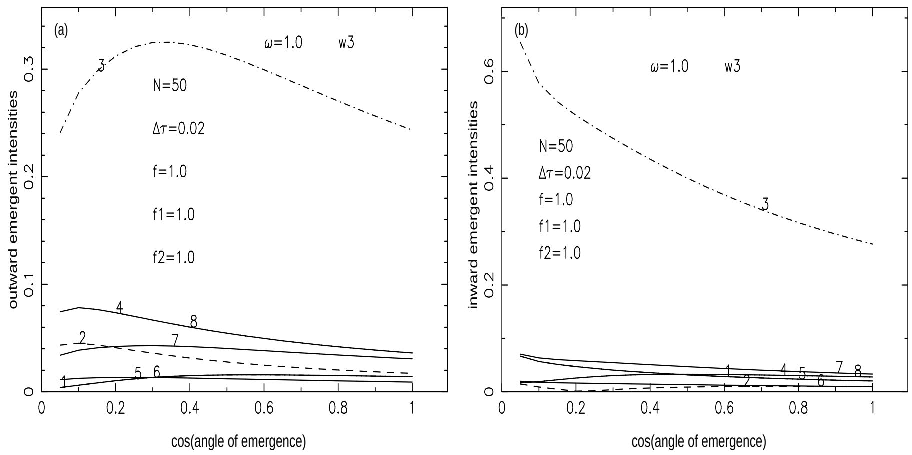

Figure 4. Outward (left) and inward (right) intensities.

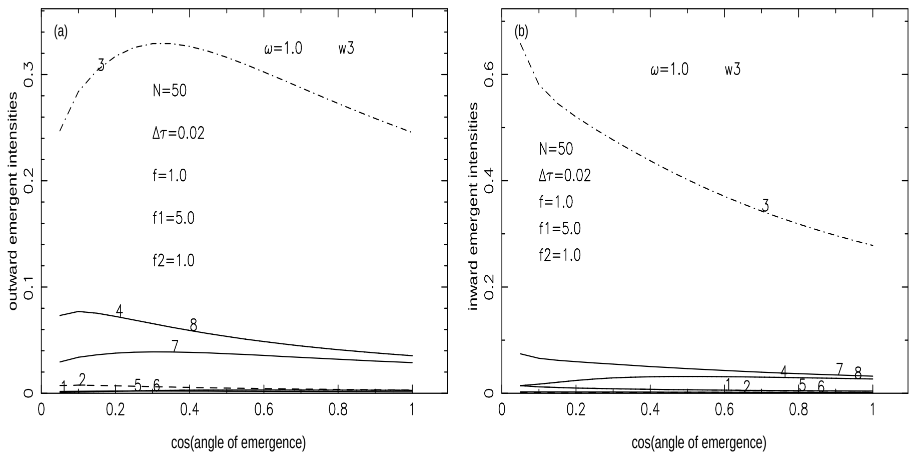

Figure 5. Outward (left) and inward (right) intensities.

which is a purely scattering case. In this case, we test the solution for radiant flux conservation. We have considered media in which both scattering and absorption operate. Here, we set,

$$\mathbf{U}_{\mathbf{1}}^{-}((w1 - w2), \tau = \tau_{max}) = 0 \tag{73}$$

$$\omega < 1 \tag{74}$$

$$B^{+}, B^{-} > 0 \tag{75}$$

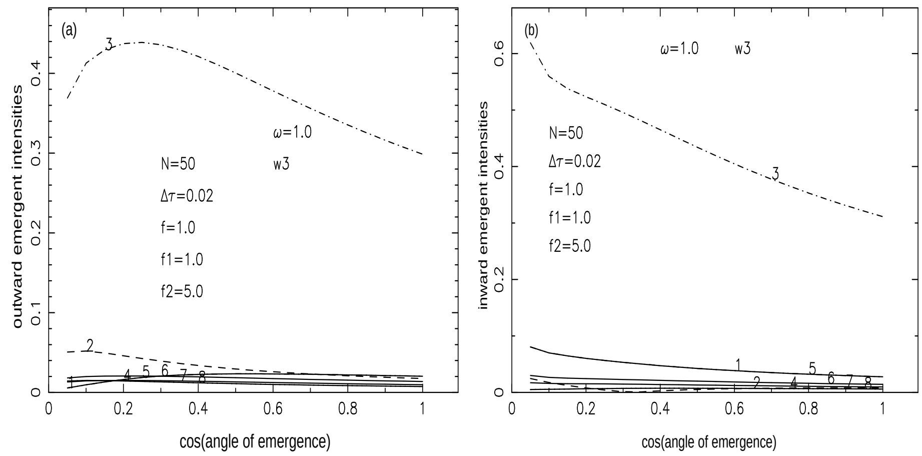

Figure 6. Outward (left) and inward (right) intensities.

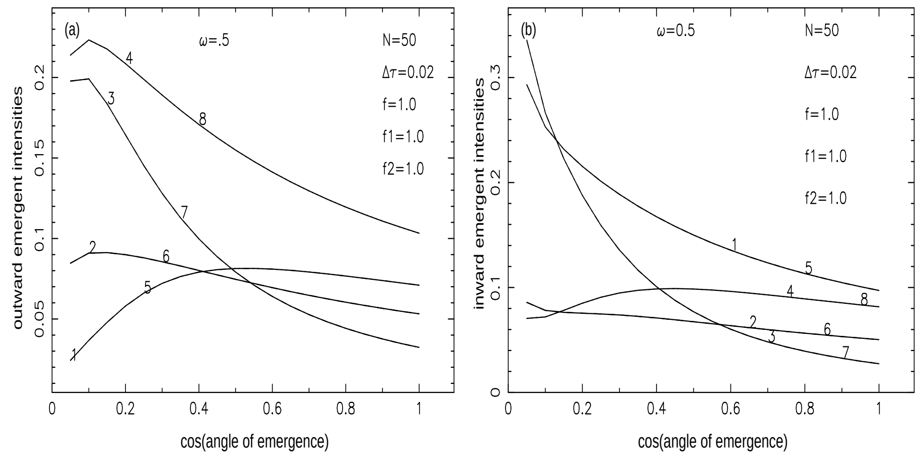

Figure 7. outward (left) and inward (right) intensities

In the calculations we have considered $\omega = 1$ and 0.5 and correspondingly we have set B=0 and 1. We have divided the medium into 50 shells each with an optical depth of 0.02 so that the total optical depth will be 1. As we noticed earlier there are two types of optical depths one along the X-axis and the other along the Y-axis (see equations (57) to (62)) $\Delta\tau_x$ and $\Delta\tau_y$ respectively and are connected through $\Delta\tau$ and the factors f,f_1 and f_2. These are the free parameters which will decide the characteristics of the medium. We have set $f_1 = 1, 5$ and $f_2 = 1, 5$ to represent different optical depths along the two axes. We have assumed isotropic phase function and therefore the elements of p are all unity each. For different values of f's the outward and the inward intensities are described in Figs (1) to (9). All the figures contain the data, that are used to calculate the results, in the figures themselves. Each figure contains two parts: (1) part (a) and (2) part (b)

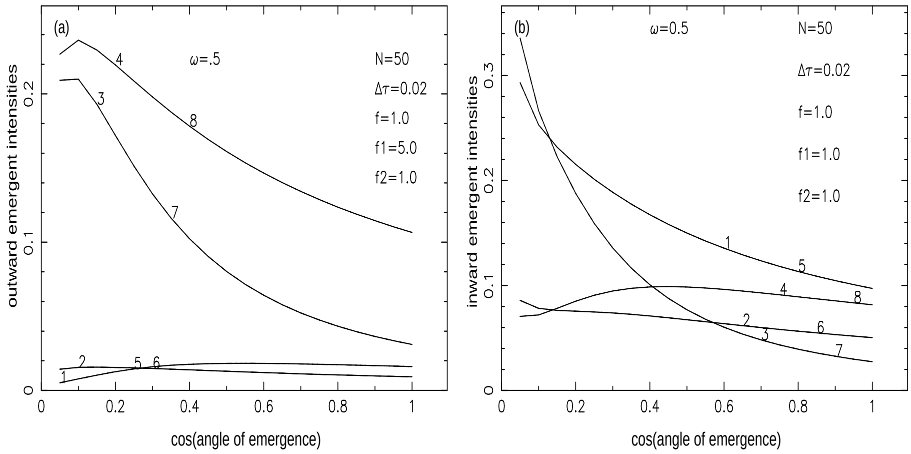

Figure 8. outward (left) and inward (right) intensities

showing the outward and the inward directed intensities respectively. The numbers on the curves represent the window number. In Fig. (1), we have used $f = f_1 = f_2 = 1$ which implies that the optical depths along X- and Y- directions are same and incident radiation is given uniformly through all the windows (see equation (72)). The outward emergent radiation and the inward directed emergent radiation are shown in this figure. Same amount of radiation emerges through windows 1 and 5. Similarly radiation emerges out through windows2 and 6, windows 3 and 7 and windows 4 and 8. This can be understood from the symmetrical nature in equations (12) to (27). If the boundary conditions are symmetric, then it is enough if we solve half the number of equations. However, when the boundary conditions are asymmetric we need to solve fully all the equations, the solutions of which are not symmetric, which we shall see below. Maximum amount of radiation seems to emerge out of the w2 (window 2) and w6 while least is emerging out of the w1 and w5 in the case of outward emerging radiation. In the inward emerging situation maximum amount of radiation emerges out of w3 and w7 while w4 and w8 releases minimum amount of radiation. It is evident that although symmetric boundary conditions are given in all the windows the emergent radiation is asymmetric. This is clearly the effect of the geometry. In Fig. 2, we changed the optical depths in the two directions that is, we set $f = 1$, $f_1 = 5$ and $f_2 = 1$. From equations (58) and (59) the optical depth along the Y-direction is $\frac{1}{5}$ of that along X-direction. The results are totally different from those given in Fig. 1. Maximum amount of radiation emerges through windows 4 and 8 both in the outward and inward directions while very little radiation emerges through windows 1 and 5 in both cases. Windows (2, 6) and (3, 7) release intermediate amount of radiation. However one should realise that the symmetry in the windows is maintained. In Fig. 3, we have presented results for $f = f_1 = 1$ and $f_2 = 5$. This implies that the optical depth along x-direction is $\frac{1}{2}$ of that along y-direction. The results are different from those given in Figs 1 and 2. More radiation emerges out of windows (2, 6) and (3, 7) while other windows emit comparatively less amount of radiation. In Fig. 4 we give the incident radiation through only one window that is w3 with $f = f_1 = f_2 = 1$. Most of the radiation emerges out of window 3 both in the outward as well as inward directions. More radiation gets back scattered into window 3. Very little amount of radiation emerges out of other windows. It is clear that although we have given incident radiation through a single window, part of this radiation gets scattered through other windows as well. Figs 5 and 6 give similar results for different values of f's (shown in the figures). We introduced absorption together with scattering in the atmosphere and the results are presented in Figs 7, 8 and 9. We set $\omega = 0.5$ and $B = 1$. We have not given incident radiation on either side of the atmosphere. Figure 7 gives the results for $f = f_1 = f_2 = 1$ which implies that the optical depths along X- and Y- axes are same. There is more outward directed radiation through w4 and w8 than through other windows. In the case of inward directed radiation more of it is coming out of windows 1 and 5 than through the other windows. Figure 8 gives the results for the data f=1,$f_1 = 5$ and $f_2 = 1$. Substantial amount of radiation is emitted in the outward direction through windows 4, 8, 3 and 7. Other windows do not contribute much to the outward directed radiation. A similar behavior is observed in the case of inward directed radiation. Fig. 9 gives the results for $f = f_1 = 1$ and $f2 = 5$. In the case of outward directed radiation windows 1,5 and 2,6 emit substantial amount of radiation while other windows do not emit as much. Same pattern of emission of inward radiation is noted.

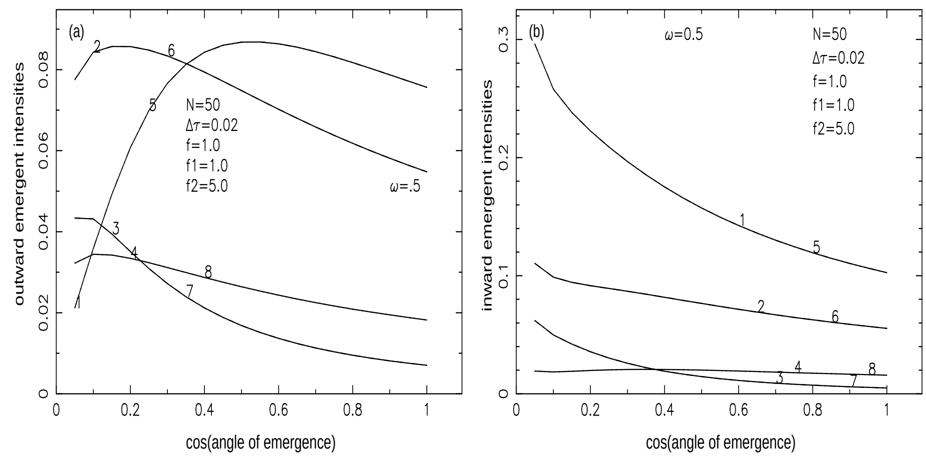

Figure 9. Outward (left) and inward (right) intensities.

4. Conclusion

The solution of transfer equation in two dimensional geometry is developed. The solution has been applied to media with different physical conditions. We have demonstrated that asymmetric boundary conditions can be used quite conveniently in an inhomogeneous atmosphere.

5. Appendix

We set,

$$
\begin{aligned}
s &= \frac{\Delta x \Delta y \sigma}{4}, t = s\omega, v = \Delta x \Delta y \sigma (1-\omega), \\
p_1 &= p^{++}c, p_2 = p^{+-}c, p_3 = p^{-+}c, p_4 = p^{--}c
\end{aligned} \tag{76}
$$

where p's are the phase functions and p^{++} means $P(+\mu, +\mu')$. Similarly other p's are defined. We give g's below:

$$g_1 = \mu\Delta y + s - tp_1, g_2 = \eta\Delta x + s - tp_1, g_3 = -tp_2, g_4 = vB^+_{m+\frac{1}{2},n+\frac{1}{2}} \tag{77}$$

$$g_5 = \mu\Delta y - s + tp_1, g_6 = \eta\Delta x - s + tp_1, g_7 = -g_3, g_8 = -tp_3 \tag{78}$$

$$g_9 = \mu\Delta y + s - tp_4, g_{10} = -\eta\Delta x + s - tp_4, g_{11} = vB^-_{m+\frac{1}{2},n+\frac{1}{2}}, g_{12} = -g_8 \tag{79}$$

$$g_{13} = \mu\Delta y - s + tp_4, g_{14} = -\eta\Delta x - s + tp_4, g_{15} = g_1, g_{16} = -\eta\Delta x + s - tp_1 \tag{80}$$

$$g_{17} = g_3, g_{18} = vB^+_{m+\frac{1}{2},-n-\frac{1}{2}}, g_{19} = g_5, g_{20} = -\eta\Delta x - s + tp_1 \tag{81}$$

$$g_{21} = -g_3, g_{22} = g_8, g_{23} = g_9, g_{24} = \eta\Delta x + s - tp_4, g_{25} = vB^-_{m+\frac{1}{2},-n-\frac{1}{2}} \tag{82}$$

$$g_{26} = -g_8, g_{27} = g_{13}, g_{28} = \eta\Delta x - s + tp_4, g_{29} = -\eta\Delta y + s - tp_1 \tag{83}$$

$$g_{30} = -\mu\Delta x + s - tp_1, g_{31} = g_3, g_{32} = vB^+_{m+\frac{1}{2},-n-\frac{1}{2}}, g_{33} = -\eta\Delta y - s + tp_1 \tag{84}$$

$$g_{34} = -\mu\Delta x - s + tp_1, g_{35} = -g_3, g_{36} = g_8, g_{37} = -\eta\Delta y + s - tp_4 \tag{85}$$

$$g_{38} = \mu\Delta x + s - tp_4, g_{39} = g_{25}, g_{40} = -g8g_{41} = -\eta\Delta y - s + tp_4 \tag{86}$$

$$g_{42} = \mu\Delta x - s + tp_4, g_{43} = \eta\Delta y + s - tp_1, g_{44} = g_{30}, g_{45} = g_3 \tag{87}$$

$$g_{46} = vB^+_{-m-\frac{1}{2},-n-\frac{1}{2}}, g_{47} = \eta\Delta y - s + tp_1, g_{48} = g_{34}, g_{49} = -g_3, g_{50} = g_8 \tag{88}$$

$$g_{51} = \eta\Delta y + s - tp_4, g_{52} = \mu\Delta x + s - tp_4, g_{53} = vB^-_{-m-\frac{1}{2},-n-\frac{1}{2}}, g_{54} = -g_8 \tag{89}$$

$$g_{55} = \eta\Delta y - s + tp_4, g_{56} = \mu\Delta x - s + tp_4, g_{57} = g_1, g_{58} = g_2, g_{59} = g_3 \tag{90}$$

$$g_{60} = g_{46}, g_{61} = g_5, g_{62} = g_6, g_{63} = -g_3, g_{64} = g_8, g_{65} = g_9 \tag{91}$$

$$g_{66} = -\eta\Delta x + s - tp_4 \tag{92}$$

$$g_{67} = g_{53}, g_{68} = -g_8, g_{69} = g_{13}, g_{70} = -\eta\Delta x - s + tp_4, g_{71} = g_1, g_{72} = g_{16} \tag{93}$$

$$g_{73} = g_3, g_{74} = vB^+_{-m-\frac{1}{2},n+\frac{1}{2}}, g_{75} = \mu\Delta y - s + tp_1, g_{76} = g_{20}, g_{77} = -g_3 \tag{94}$$

$$g_{78} = g_8, g_{79} = g_9, g_{80} = \eta\Delta x - s - tp_4, g_{81} = vB^-_{-m-\frac{1}{2},n+\frac{1}{2}}, g_{82} = -g_8 \tag{95}$$

$$g_{83} = g_{13}, g_{84} = \eta\Delta x - s + tp_4, g_{85} = g_{29}, g_{86} = g_{30}, g_{87} = g_3, g_{88} = g_{74} \tag{96}$$

$$g_{89} = g_{33}, g_{90} = g_{34}, g_{91} = -g_3, g_{92} = g_8, g_{93} = g_{37}, g_{94} = g_{38}, g_{95} = g_{81} \tag{97}$$

$$g_{96} = -g_8, g_{97} = g_{41}, g_{98} = g_{42}, g_{99} = g_{43}, g_{100} = g_{30}, g_{101} = g_3, g_{102} = g_4 \tag{98}$$

$$g_{103} = \eta\Delta y - s + tp_1, g_{104} = g_{34}, g_{105} = -g_3, g_{106} = g_8, g_{107} = g_{51} \tag{99}$$

$$g_{108} = g_{38}, g_{109} = g_{11}, g_{110} = -g_8, g_{111} = g_{55}, g_{112} = g_{42} \tag{100}$$

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