Bull. Astr. Soc. India (2007) **35**, 1–13

The nonlinear thermal instability in ISM: typical net cooling functions

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Received 30 September 2006; accepted 12 January 2007

Abstract. The nonlinear evolution of thermally unstable disturbances in plasma are investigated within the framework of one-dimensional fluid equations. The processes of heating and radiative cooling (cooling function) of the optically thin plasma are taken into account. A new statistical approach is presented to study the thermal instability to the interstellar medium ISM of an optically thin unmagnetized plasma. This approach makes possible to simplify the set of equations describing the instability, and using Lagrangian coordinates, to investigate the nonlinear dynamics of the instability analytically. The equations are solved both analytically and numerically by successive approximations assuming a thermal conduction coefficient T and cooling function coefficient of type $T^a - T^b$. We discuss the nonlinear development of the isobaric mode of thermal instability in the atomic molecular clouds of ISM.

Keywords : Instability: general - ISM

1. Introduction

One of the most important and interesting dynamical processes in astrophysical plasma and gases which are subject to some external heating and radiative cooling is the thermal instability (Parker 1953; Field 1965). There are many physical situations where a steady temperature distribution is maintained by means of basic energy transportation mechanisms: heat diffusion, generation of heat into, and radiation of energy from a particular configuration. Most theoretical efforts aimed at studying thermal instability were limited

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to the linear theory, treating unstable perturbations as small compared to the "unperturbed" quantities. A thorough analysis of this linear stage is the paper by Field (1965). The nonlinear stage of the instability has been investigated in many papers numerically (Goldsmith 1970; Schwarz 1972; Schmack & VanHoven 1991). In recent years, there has been an increasing interest and activity in the investigation of the dynamics of coherent structures and pattern formation in macroscopic systems. New methods of nonlinear dynamics, Spatial chaos, and pattern theory have been developed by (Bishop, Campbell & Channel 1984; Busse & Kramer 1990) and for general introduction to the methods of nonlinear dynamics (Tabor 1989). The structure of the paper is as follow: In section 2, we drive a set of simplified nonlinear equations. In section 3, we make a transition from the Eulerain to Lagrangian coordinates and obtain a single nonlinear partial differential equation for the evolution of the temperature of the unstable gas. In section 4, we briefly analyze the limits corresponding to very small value of the thermal conductivity parameter. In section 5, we briefly analyze the possible steady state of the gas. Finally, in section 6, the mentioned consideration will be generalized and solved in nonlinear regime, analytically by ordinary differential equation (ODE).

2. The basic equations

Let us consider a one-dimensional flow of an ideal optically thin gas of density ρ, Temperature T and velocity v under the action of a gradient of the pressure P (gravity , magnetic field and other forces are neglected). The governing fluid equations have their usual form (Field 1965):

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla} \cdot (\rho \overrightarrow{v}) = 0 \tag{1}$$

$$\rho \frac{D\overrightarrow{v}}{Dt} + \overrightarrow{\nabla} \cdot \overrightarrow{P} = 0 \tag{2}$$

$$\frac{R}{\mu} \rho T \frac{Ds}{Dt} = -\rho L(\rho, T) + \overrightarrow{\nabla} \cdot (K \overrightarrow{\nabla} T) \tag{3}$$

$$P = \frac{R}{\mu} \rho t \tag{4}$$

where $\frac{D}{Dt}$ is the Lagrangian derivative. Here $L(\rho, T)$ is the heat-loss function (the difference between the rate of the radiative cooling and the rate of heating per unit mass), $K(T)$ is the thermal conductivity, μ is its effective molar mass, and R is the gas constant. The heat-loss function is determined by the specific mechanisms of heating and cooling. In the case of interstellar medium, the heating rate is determined by the absorption UV and soft X-rays or cosmic rays, while radiative cooling rate is determined by various processes (such as the inverse Compton cooling, Bremsstrahlung, atomic and molecular processes and grain cooling) depending on the temperature and density of the gas (for details Pikelner 1979 ; Lepp 1985). Usually this function is written as,

$$L(\rho, t) = \rho L(T) - G(\rho, T) \tag{5}$$

where G and ρL the heating and cooling rate per unit mass. Field (1965) investigated the initial, linear stage of this thermal runaway by means of the linearization of equations (1)-(4). The linear theory of the instability predicts the exponential growth of small initial perturbations. The necessary condition for the instability of the condensation mode obtained by Field has the form below,

$$\left(\frac{\partial L}{\partial T}\right)_p = \left(\frac{\partial L}{\partial T}\right)_\rho - \frac{\rho_o}{T_o}\left(\frac{\partial L}{\partial P}\right)_T < 0 \tag{6}$$

To give a complete description of the nonlinear thermal runaway, one needs to solve nonlinear fluid equations mentioned before. Being interested in the development of the thermal instability in the interstellar medium, we put $P = const$ and obtain the following set of simplified equations

$$\frac{R}{\mu}(\frac{\gamma-1}{\gamma})T_o\rho_o\frac{\partial v}{\partial x} + \rho L(\rho,T) - \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) = 0 \tag{7}$$

$$\rho T = \rho_o T_o = const \tag{8}$$

where γ is specific heat ratio of gas. Condition(8) makes it possible to eliminate the density from equations (1) and (7). Now equations (1)-(4) can be written in a nondimensional form if all the physical variables are scaled their representative values (Meerson 1989)

$$\frac{\partial T}{\partial t} - T^2\frac{\partial}{\partial x}(\frac{v}{T}) = 0 \tag{9}$$

$$T\frac{\partial v}{\partial x} + \lambda(T) - k_o T\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) = 0 \tag{10}$$

where $\lambda(T)$ is the dimensionless heat-loss function. Dimensionless equations (9)-(10) describe the nonlinear evolution of the isobaric condensation mode. By solving equations (9) and (10), we shall be able to determine the dimensionless density $\rho(x,T)$. Now Let us proceed to the analysis of equations (9) and (10).

3. Equations in Lagrangian coordinates

It is well known that one of the most efficient methods of solving nonlinear fluid equations, especially in the case of one-dimensional flows is the transition to Lagrangian coordinates (Zeldovich 1967). It is convenient to introduce the Lagrangian mass variable

$$m = \int_{x_1(t)}^{x}\rho(x,t)dx = \int_{x_1(t)}^{x}T^{-1}(x,t)dx \tag{11}$$

where for all t the coordinate of a reference "particle" $x_1(t)$ is determined by the condition $v(x_1,t) = 0$. Now let us transfer from variables x and t to new variables m and t. The continuity equation (9) will assume a simple form

$$\frac{\partial T}{\partial t} = \frac{\partial v}{\partial m} \tag{12}$$

While the isobaric thermal balance equation (10) will be as follows

$$\frac{\partial v}{\partial m} + \lambda(T) - \frac{\partial}{\partial m}\left(\frac{K(T)}{T}\frac{\partial T}{\partial m}\right) = 0. \tag{13}$$

From equations (12) and (13) we have

$$\frac{\partial T}{\partial t} + \lambda(T) - \frac{\partial}{\partial m}\left(\frac{K(T)}{T}\frac{\partial T}{\partial m}\right) = 0 \tag{14}$$

$$\frac{\partial T}{\partial t} + \lambda(T) - k_o T\frac{\partial}{\partial x}(K(T)\frac{\partial T}{\partial x}) = 0. \tag{15}$$

Thus we have obtained a single nonlinear partial differential equation which describes the evolution of the unstable gas temperature in the Lagrangian coordinates. If we succeed in solving equations (14) and (15) if we find the temperature $T(m,t)$ we can easily determine the rest of the variables. The gas density is determined simply by

$$\rho(m,t) = \frac{1}{T(m,t)} \tag{16}$$

4. Solution in limit of low thermal conductivity

This limit corresponds to very small values of the parameter $k_o \to 0$. Thus we obtain a simple equation

$$\frac{\partial T(m,t)}{\partial t} + \lambda(T) = 0 \tag{17}$$

which can be integrated to give

$$t = \int_T^{T_o} \frac{dT}{\lambda(T)} \tag{18}$$

where $T_o(m) = T(m, t = 0)$ is the initial form of the temperature perturbation in the Lagrangian coordinates. In many cases we can model the heat-loss function $\lambda(T)$ by a difference of two powers (Meerson 1989)

$$\lambda(T) = T^a - T^b \tag{19}$$

where the powers a and b can take any real values. The equilibrium point $T = 1$, described by equations (17) and (19), can be easily seen to be unstable if $a < b$. It can also be seen that two types of singularities may develop in an unstable gas described by an idealized heating-cooling curve. The case $a = 1, b = 2$ is especially simple,

$$\lambda(T) = T(1 - T) \tag{20}$$

and it may be regarded as partially degenerate, because equation (17) becomes linear

$$\frac{dT}{dt} + T(1 - T) = 0 \tag{21}$$

Here we have an unstable equilibrium point $T = 1$ and a stable point $T = 0$. The temperature and density of the gas is determined from equation (18)

$$T(m,t) = \left[1 + \left(\frac{1 - T_o(m)}{T_o(m)}\right) e^t\right]^{-1} \tag{22}$$

$$\rho(m,t) = 1 + \left(\frac{1 - T_o(m)}{T_o(m)}\right) e^t. \tag{23}$$

It is clear from equation (22) that if we start from a perturbation, $\delta T(m) = T_o(m) - 1$ the regions with $\delta T(m) > 0$ will be heated and by choosing $\delta T(m) < 0$ that region will be cooled. In this case, the heating rate of regions with $\delta T(m) > 0$ increase rapidly with the temperature. Therefore the singularity of another type, $T \to \infty$, will arise at a point m^* i.e.

$$t^* = Ln\left[\frac{T_o(m^*)}{T_o(m^*) - 1}\right]. \tag{24}$$

Another simple example, describing the same type of singularity, is provided by a localized initial temperature perturbation

$$T_o(m) = \frac{1 + m^2}{1 + m^2 - \alpha}, \qquad 0 < \alpha < 1. \tag{25}$$

Using formulae (22),(23) and (24), we obtain the following expressions for the variables Lagrangian coordinates

$$T(m,t) = \frac{1 + m^2}{1 + m^2 - \beta} \tag{26}$$

$$\rho = \frac{1 + m^2 - \beta}{1 + m^2} \tag{27}$$

$$x(m,t) = m + \frac{\beta}{\sqrt{1 - \beta}} \tan^{-1} \frac{m}{\sqrt{1 - \beta}} \tag{28}$$

where $\beta = \alpha e^t$. Fig.1 shows the temporal evolution of the flow variables in this case. Another simple example assumes $T_o(m) = \frac{1}{1 - \alpha \cos(m)}$, using formulae (22) and (23) we obtain the following expressions for the flow variable in the Lagrangian coordinates

$$T(m,t) = (1 - \beta \cos(m))^{-1} \tag{29}$$

$$\rho(m,t) = (1 - \beta \cos(m)) \tag{30}$$

where $\beta = \alpha e^t$, in the case of $\lambda(T) = T(1 - T^2)$, The temperature of the gas is determined by equation (17)

$$T(m,t) = \left(1 + \left(\frac{1}{T_o^2(m)} - 1\right) e^{2t}\right)^{-\frac{1}{2}} \tag{31}$$

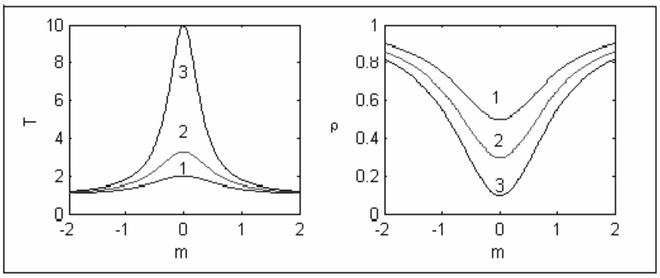

Figure 1. Evolution of a localized initial perturbation(eq. [25]). The temperature and density profiles are shown thrice, corresponding to, $\beta = 0.9$, (curve 3), $\beta = 0.7$(curve 2), $\beta = 0.5$ (curve 1).

and density of gas is given by

$$\rho(m,t) = \left(1 + \left(\frac{1}{T_o^2(m)} - 1\right) e^{2t}\right)^{\frac{1}{2}} \tag{32}$$

where we have consider a perturbation in the form of, $\delta T(m) = T_o(m)^2 - 1$. The regions with $T_o^2(m) > 1$ will be heated and become infinite at m^* at time,

$$t^* = \frac{1}{2} \ln \left(\frac{T_o^2(m^*)}{T_o^2(m^*) - 1} \right) \tag{33}$$

Another example using initial Temperature

$$T_o(m) = \left(\frac{1}{1 + m^2 - \epsilon}\right)^{\frac{1}{2}}, \qquad 0 < \epsilon < 1 \tag{34}$$

using formulae (31) and (32) will give,

$$T(m,t) = \left(1 + \left(m^2 - \epsilon\right) e^{2t}\right)^{-\frac{1}{2}} \tag{35}$$

$$\rho(m,t) = \left(1 + \left(m^2 - \epsilon\right) e^{2t}\right)^{\frac{1}{2}} \tag{36}$$

Fig. 2 shows the temporal evolution of the flow variables in this case.

5. Thermal instability of the steady solution

In many cases that we have considered briefly, the heat conduction term and heat-loss function in equations (14) and (15) play an important role in determining the nonlinear dynamics of the instability and possible steady state of the gas. If the conductivity,

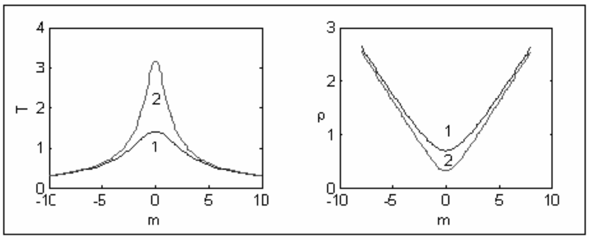

Figure 2. Evolution of a localized initial perturbation(eq. [34]). The temperature and density profiles are shown twice, corresponding to, $1-\epsilon e^{2t} = 0.5, \epsilon e^{t} = 0.1$, (curve 1), $1-\epsilon e^{2t} = 0.1, \epsilon e^{t} = 0.1$ (curve 2).

$K(T)$, is typically a function of T, the power law $K \propto T^n$ covers most cases of interest. Zeldovich and Pikelner (1969) adopted the value $n = \frac{1}{2}$, appropriate when the thermal conductivity is determined by the neutral diffusion. We introduce the variable $u = T^{n+1}$, the situation satisfies equation (15)

$$\frac{\partial u}{\partial t} = u\nabla^2 u - L(u,p) \qquad (37)$$

where $L(u,p) = -(1+n)T^n\lambda(T,p)$ and $p = const$. The steady states are described by the equation (one- dimensional)

$$\frac{d^2u}{dx^2} - L(u) = 0 \qquad (38)$$

if we chose $L(u) = e^{-u}$, equation (38) can be simplified to

$$\frac{d^2u}{dx^2} - e^{-u} = 0. \qquad (39)$$

We introduce the variable $e^{-u} = w$, under transformation to this new independent variable we obtain

$$\frac{dw}{dx} = e^{-u}\left(-\frac{du}{dx}\right) = -w\frac{du}{dx} \qquad (40)$$

$$\frac{d}{dx}\left(\frac{dw}{dx}\right) = -\frac{dw}{dx}\frac{du}{dx} - w\frac{d^2u}{dx^2} \Rightarrow \frac{d^2w}{dx^2} = \frac{1}{w}\left(\frac{dw}{dx}\right)^2 - w^2. \qquad (41)$$

Substituting (39) in the equation (41), we obtain

$$w\frac{d^2w}{dx^2} - \left(\frac{dw}{dx}\right)^2 + w^3 = 0 \qquad (42)$$

If we assume

$$\frac{dw}{dx} = f(w) \qquad (43)$$

we have

$$\frac{d^2w}{dx^2} = \frac{1}{2}\frac{df^2}{dw}. \tag{44}$$

Using (42),(43)and (44) we obtain the following equation

$$\frac{w}{2}\frac{df^2}{dw} - f^2 + w^3 = 0 \tag{45}$$

Seeking a solution of the series form

$$f^2 = \sum_{n=0}^{\infty} a_n w^n \Rightarrow \frac{df^2}{dw} = \sum_{n=1}^{\infty} n a_n w^{n-1} \tag{46}$$

and substituting (46) in the (45) leading to

$$a_o = const, \quad a_1 = 0, \quad a_2 = const = k, \quad a_3 = -2, \quad a_4 = a_5 = a_6 = ... = 0 \tag{47}$$

from (47) and (48) we obtained

$$f^2(w) = kw^2 - 2w^3, \tag{48}$$

and from (43) and (48) obtained

$$x = \int \frac{dw}{w\sqrt{a - 2w}} \tag{49}$$

$$x = \frac{2}{\sqrt{a}} \arccos\left(\sqrt{\frac{a}{2w}})\right) \tag{50}$$

where $k = a = const \neq 0$, now from $u = \ln \frac{1}{w}$ and (50), we obtained the following

$$u = \ln\left(\frac{2\cos\left(\frac{x\sqrt{a}}{2}\right)}{a}\right) \tag{51}$$

Fig.3, shows the u evolution of x.

6. Nonlinear instability analysis to trivial solution

Generally, for static configuration, pressure p is constant with respect to time, and therefore the equation of conservation of energy can be written as (Elphick 1992)

$$\frac{\partial T}{\partial t} = -\lambda(T) + \frac{\partial}{\partial m}\left(\frac{K(T)}{T}\frac{\partial T}{\partial m}\right). \tag{52}$$

In many physical situations the quantities $K(T)$ can be written as a first approximation, in the form below,

$$K(T) = k_o T^{\alpha} \tag{53}$$

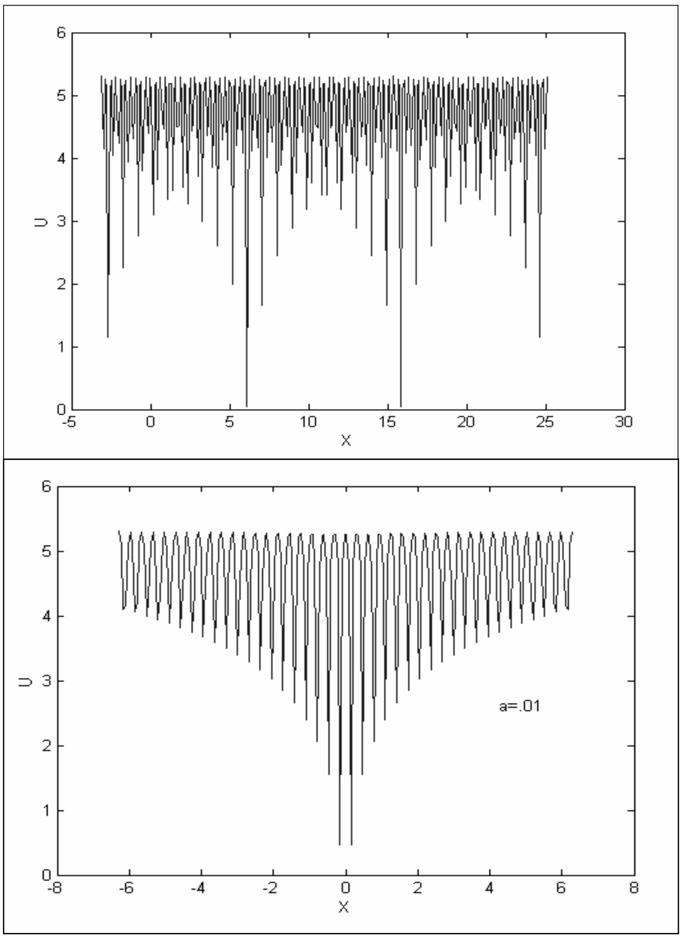

Figure 3. Evolution of the temperature $u = T^{n+1}$ of the unstable gas (eq. [51]): ($a = 0.01$).

where k_o and α are known constants. For example, for thermal conduction by neutral particles $\alpha = \frac{1}{2}$ (Parker 1953). For radiation thermal condition $\alpha = \frac{13}{2}$ (if $\rho = const$). For $K(T) = T$ and $\lambda(T) = 1 - T$ equation (52) can be written in the form as follows,

$$\frac{\partial T}{\partial t} = T - 1 + \frac{\partial^2 T}{\partial m^2} \tag{54}$$

where $\alpha = 1$ and $k_o = 1$. Assuming a solution for equation (54) of the form

$$T(m,t) = T_o + g(m)e^{kt} \tag{55}$$

where T_o is the background temperature of the medium and k is constant, one obtains

$$g(m) = A\cos(wm) + B\sin(wm) + \frac{(1-T_o)e^{-kt}}{w} \tag{56}$$

with the corresponding boundary conditions it will be assumed to be

$$\begin{array}{llll} m & = 0 & at & g(m) & = 1 \\ m & = 0 & at & \frac{dg}{dm} & = 0. \end{array} \tag{57}$$

The equations (54) and (55) with boundary condition (57) has the solutions

$$T(m,t) = T_o + e^{kt} - \frac{1-T_o}{w}\cos(wm) + \frac{1-T_o}{w} \tag{58}$$

where $w = \sqrt{1-k}$ and $w > 0$. Fig.4 shows the temporal evolution.

In many physical situations the conductivity temperature $K(T)$ can be written in the form below, (Ibanez & Plachco 1991).

$$K(T) = k_o T^a \tag{59}$$

where k_o and a being known constants. Therefore equation (52) can be simplified to

$$\frac{\partial y}{\partial t} = F(y) + y^b \frac{\partial^2 y}{\partial m^2} \tag{60}$$

where $k_o = 1$ and $y = T^a$, $F(y) = aT^{ba}\lambda(T)$, $b = \frac{a-1}{a}$.
We can model the heat-loss function by a difference of two power $a = 2, b = \frac{1}{2}$.

$$f(y) = \sqrt{y} - 1 \tag{61}$$

using Eq.(61) and defining, $\xi \equiv m - \alpha t$, with $\alpha = const$, can be equation (61) as,

$$y'' + \alpha\frac{y'}{\sqrt{y}} - \frac{1}{\sqrt{y}} + 1 = 0 \tag{62}$$

where $y' = \frac{dy}{d\xi}$. We shall call this equation the associated ODE (ordinary differential equation). For the given autonomous equation, put $y' = p$, it will become

$$\frac{dp}{dy} = \left(\frac{-\alpha p - \sqrt{y} + 1}{p\sqrt{y}}\right) \tag{63}$$

Now let $z = \sqrt{y} - 1$ in (63)to get

$$\frac{dp}{dz} = -2\left(\frac{z + \alpha p}{p}\right). \tag{64}$$

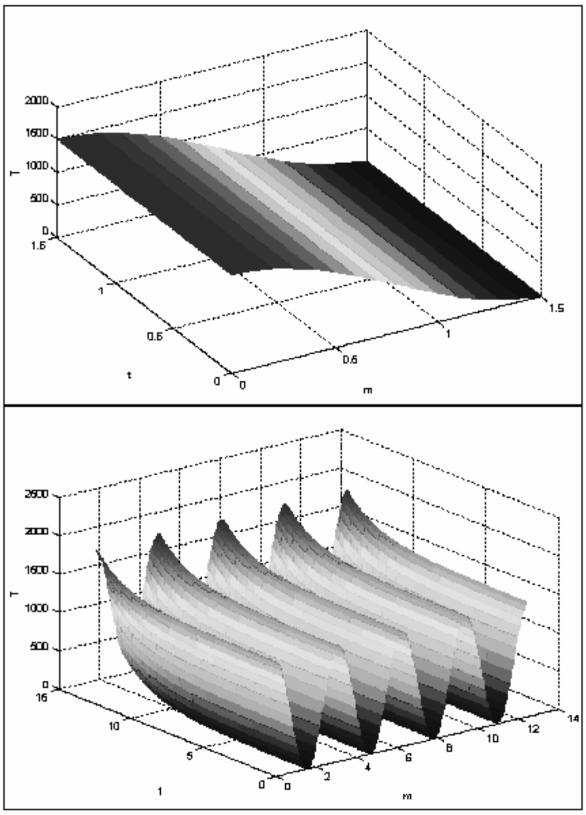

Figure 4. The evolution of temperature for the unstable gas (eq. [58]), the line shows the evolution coming from nonlinear theory, $T_o = 1500, w = 2$.

Equation (63) can be easily integrated by writing

$$-\ln z + c = \int \frac{wdw}{w^2 + 2\alpha w + 2} \tag{65}$$

where $c = const$. The second integral ($\int \frac{dw}{w^2+2\alpha w+2}$) is found to be

$$\begin{aligned} \int \frac{dw}{(w^2 + 2\alpha w + 2)} &= \left(\frac{1}{2-\alpha^2}\right)^{\frac{-1}{2}} \tan^{-1}\left[\frac{w+1}{(2-\alpha^2)}\right], \quad && \alpha^2 < 2 \\ -\frac{1}{w+\alpha}, & && \alpha^2 = 2 \\ \frac{1}{(\alpha^2-2)^{\frac{1}{2}}} \ln\left[\frac{w+\alpha-(\alpha^2-2)^{\frac{1}{2}}}{w+\alpha+(\alpha^2-2)^{\frac{1}{2}}}\right], & && \alpha^2 > 2 \end{aligned} \tag{66}$$

The initial conditions corresponding to equation (62)will be taken as

$$y(0) = y'(0) = 0 \tag{67}$$

equation (62) with conditions (67) can be integrated to compute c,

$$c = \frac{1}{2}\ln 2 - \frac{\alpha}{\sqrt{2-\alpha^2}}\tan^{-1}\left(\frac{\alpha}{\sqrt{2-\alpha^2}}\right), \qquad \alpha^2 < 2 \tag{68}$$

The intermediate integrals (66) and (65) do not admit further solution. The Taylor series solutions for $\alpha = 1$ about their first maxima are

$$\begin{aligned} y(\xi) = 1.08 - 2.07\times10^{-2}(\xi-\xi_o)^2 + 6.6\times10^{-3}(\xi-\xi_o)^3 \\ - 8.25\times10^{-4}(\xi-\xi_o)^4 + 3.14\times10^{-5}(\xi-\xi_o)^5 \\ -1.37\times10^{-6}(\xi-\xi_o)^6 + 8.3\times^{-7}(\xi-\xi_o)^7 - 5.6\times^{-8}(\xi-\xi_o)^8 \end{aligned} \tag{69}$$

where $\xi_o = \xi(m, t = o)$. In this solution as we can see in spite of different values of ξ and the first condition ξ_o, the medium will be hot or cool. For example when $0 < \xi < 1$ and $\xi < \xi_o < 0$ the medium will become slowly cool and dense, but by the condition $0 < \xi < 1$, $0 < \xi < \xi_o$, the medium becomes cool with more speed than the first condition.

7. Conclusions

In this paper, we have studied the dynamics of thermal conduction front in the interstellar medium with parameters suitable to describe the general (ISM). The inclusion of radiative losses affects both the dynamics and the structure of the conductive/cooling front. The results of the present study shows that in the very small values of the parameter k_o, the "effective pressure" concept is inadequate. Instead, a consistent analysis requires the use of the isobaric heating-cooling function (Lepp 1985), which provides a full description of these limiting cases. The steady solution were obtained by analytical solving of equation (14) and they were classified by means of two parameters: thermal conductivity and heat-loss function. Finally, trivial solution and values of a, b, and α are fixed, the dimensions of marginally stable slabs.

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