

## Property of Tsallis entropy and principle of entropy increase

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**Abstract.** The property of Tsallis entropy is examined when considering two systems with different temperatures to be in contact with each other and to reach the thermal equilibrium. It is verified that the total Tsallis entropy of the two systems cannot decrease after the contact of the systems. We derived an inequality for the change of Tsallis entropy in such an example, which leads to a generalization of the principle of entropy increase in the framework of nonextensive statistical mechanics.

*Keywords :* nonextensive system – principle of entropy increase

### 1. Introduction

In recent years, a generalization of Boltzmann-Gibbs(B-G) statistical mechanics initiated by Tsallis has focused a great deal of attention, the results from the assumption of nonadditive statistical entropies and the maximum statistical entropy principle, which has been known as nonextensive statistical mechanics (NSM) (Abe & Okamoto 2001). This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis 1988) as follows:

$$S_q = \frac{k}{1-q} \left( \int \rho^q d\Omega - 1 \right) \quad (1)$$

where  $k$  is the Boltzmann's constant. For a classical Hamiltonian system,  $\rho$  is the phase space probability distribution of the system under consideration that satisfies the normalization  $\int \rho d\Omega = 1$  and  $d\Omega$  stands for the phase space volume element. The entropy index

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$q$  is a positive parameter whose deviation from unity is thought to describe the degree of nonextensivity. The conventional B-G entropy  $S_B$  is obtained as the limit  $q \rightarrow 1$ ,

$$S_B = \lim_{q \rightarrow 1} S_q = -k \int \rho \ln \rho d\Omega. \quad (2)$$

One distinctive difference between Tsallis entropy  $S_q$  and B-G entropy  $S_B$  lies in its pseudo-additive property. If  $a$  and  $b$  are two independent systems in the sense that the probability distribution of  $a + b$  factorizes into those of  $a$  and of  $b$ , then Tsallis entropy of the composite system  $a + b$  can be expressed by

$$S_q(a + b) = S_q(a) + S_q(b) + (1 - q)S_q(a)S_q(b)/k. \quad (3)$$

Such a generalization may address the nonextensive systems with long-range interactions, long-range microscopic memory, and fractal or multifractal relevant space-time. It retains some of the structure of the standard theory such as the Legendre transform structure,  $H$  theorem, Onsager reciprocity theorem, fluctuation-dissipation theorem, zeroth law of thermodynamics, equipartition and virial theorem etc. (please see <http://tsallis.cat.cbpf.br/biblio.htm>). Some applications of NSM in the astrophysical self-gravitating systems (Du 2006, 2007 and references therein) and in the plasma systems (Du 2004 and the references therein; Lavagno & Quarati 2006; Shaikh et al. 2007) with the long-range interactions have been investigated.

When the traditional B-G statistical mechanics is generalized, it is most important to establish consistency of the new theory with the basic principles of thermodynamics. In this paper, we study the principle of entropy increase by using Tsallis entropy in the NSM and try to give a mathematical formulation about the second law of thermodynamics in the framework of NSM.

## 2. Entropy increase in B-G statistics

In B-G statistical mechanics, the principle of entropy increase is known as a mathematical formulation of the second law of thermodynamics. It states that when a system undergoes an adiabatic process its entropy can never be decreased, the entropy is unchanged if the process is reversible, or the entropy increases if the process is irreversible. The principle of entropy increase can be verified from the canonical ensemble theory by studying changes of the entropy if two systems with different temperatures are in contact with each other and be in thermal equilibrium.

We consider two extensive systems, their equilibrium properties are determined by the canonical probability distribution  $\rho_1$  and  $\rho_2$ , and the corresponding phase space volume elements are denoted by  $d\Omega_1$  and  $d\Omega_2$ , respectively. Without the contact, the two systems are independent and are regarded as one system. The probability distribution for the total system and phase space volume elements are  $\rho_0 = \rho_1\rho_2$  and  $d\Omega = d\Omega_1d\Omega_2$ , respectively,

and the total entropy is considered in terms of the additivity as

$$S_{B0} = -k \int \rho_1 \ln \rho_1 d\Omega_1 - k \int \rho_2 \ln \rho_2 d\Omega_2 = -k \int \rho_0 \ln \rho_0 d\Omega \quad (4)$$

with  $\rho_1 = \frac{1}{Z_1} e^{-\beta_1 H_1}$  and  $\rho_2 = \frac{1}{Z_2} e^{-\beta_2 H_2}$ . The energy of every subsystem is expressed, respectively, by

$$\langle H_1 \rangle = \int \rho_1 H_1 d\Omega_1 = \int \rho_0 H_1 d\Omega; \quad \langle H_2 \rangle = \int \rho_0 H_2 d\Omega. \quad (5)$$

After the contact, the new equilibrium in the total system is determined by the canonical probability distribution  $\rho = \frac{1}{Z} e^{-\beta H}$ . The corresponding entropy is given as

$$S_B = -k \int \rho \ln \rho d\Omega. \quad (6)$$

Then, the change of B-G entropy can be calculated by

$$\begin{aligned} S_B - S_{B0} &= k \int (\rho_0 \ln \rho_0 - \rho \ln \rho) d\Omega \\ &= k \langle X(\rho_0, \rho) \rangle_0 - k \int (\rho - \rho_0) \ln \rho d\Omega \\ &= k \langle X(\rho_0, \rho) \rangle_0 - k\beta \int (\rho - \rho_0) H d\Omega \end{aligned} \quad (7)$$

where  $X(\rho_0, \rho) = \ln \rho_0 - \ln \rho$  and  $\langle X(\rho_0, \rho) \rangle_0 = \int \rho_0 X(\rho_0, \rho) d\Omega$ . It has been verified (Wang Zhuxi 1965) that the inequality  $\langle X(\rho_0, \rho) \rangle_0 \geq 0$  is always satisfied and the equality holds if and only if we have  $\rho = \rho_0$ . Suppose the interaction between the two subsystems is small so that we can write  $H = H_1 + H_2$  and, after the contact, the total energy is unchanged, then the second term on the right hand side of Eq.(7) vanishes, we have

$$S_B - S_{B0} \geq 0. \quad (8)$$

The two subsystems being in contact with each other can be regarded as an isolated system, in which any processes taking place can be treated as the adiabatic ones. This inequality shows that the entropy cannot be decreased in the adiabatic processes. The entropy is unchanged only if  $\rho = \rho_0$ , which holds only if the two systems have the same temperature:  $\beta_1 = \beta_2 = \beta$ . The inequality (8) can be thought as the mathematical expression of the principle of entropy increases in the above example.

### 3. Principle of entropy increase for the nonextensive system

Now we deal with the property of Tsallis entropy for nonextensive systems and try to give a generalization of the principle of entropy increase, Eq. (8). We consider two

nonextensive systems with different temperatures and with the Hamiltonians  $H_1$  and  $H_2$ , respectively. Their Tsallis' entropies are given, respectively, by

$$S_{q1} = \frac{k}{1-q} \left( \int \rho_1^q d\Omega_1 - 1 \right); \quad S_{q2} = \frac{k}{1-q} \left( \int \rho_2^q d\Omega_2 - 1 \right). \quad (9)$$

In NSM, the corresponding optimal probability distributions and the generalized canonical partition functions (Tsallis 1998), respectively, to be

$$\rho_1 = \frac{1}{Z_{q1}} [1 - (1-q)\beta_1^*(H_1 - U_{q1})]^{-\frac{1}{1-q}}; \quad (10)$$

$$\rho_2 = \frac{1}{Z_{q2}} [1 - (1-q)\beta_2^*(H_2 - U_{q2})]^{-\frac{1}{1-q}} \quad (11)$$

$$Z_{q1} = \int [1 - (1-q)\beta_1^*(H_1 - U_{q1})]^{-\frac{1}{1-q}} d\Omega_1 \quad (12)$$

$$Z_{q2} = \int [1 - (1-q)\beta_2^*(H_2 - U_{q2})]^{-\frac{1}{1-q}} d\Omega_2 \quad (13)$$

where  $U_{q1}$  and  $U_{q2}$  are given in terms of the constraints on the generalized internal energy  $\langle H_1 \rangle_q$  and  $\langle H_2 \rangle_q$ , defined by the normalized  $q$ -expectation values,

$$U_{q1} = \langle H_1 \rangle_q = \int \rho_1^q H_1 d\Omega_1 / \int \rho_1^q d\Omega_1, \quad (14)$$

$$U_{q2} = \langle H_2 \rangle_q = \int \rho_2^q H_2 d\Omega_2 / \int \rho_2^q d\Omega_2. \quad (15)$$

$\beta_1^* = \beta_1 / \int \rho_1^q d\Omega_1$  and  $\beta_2^* = \beta_2 / \int \rho_2^q d\Omega_2$  with  $\beta_1$  and  $\beta_2$  the Lagrange multipliers associated with the two energy constraints above, respectively.  $\beta_1$  and  $\beta_2$  are identified with their inverse temperatures.

Without the contact, the two nonextensive systems are independent. The probability distribution function and the phase space volume element of the total system are  $\rho_0 = \rho_1 \rho_2$  and  $d\Omega = d\Omega_1 d\Omega_2$ , respectively. The total energy is

$$U_{q1} + U_{q2} = \int \rho_0^q (H_1 + H_2) d\Omega / \int \rho_0^q d\Omega. \quad (16)$$

Tsallis' entropy of the total system satisfies the pseudoadditivity of Eq. (3), namely

$$S_{q0} = S_{q1} + S_{q2} + \frac{1-q}{k} S_{q1} S_{q2} = \frac{k}{1-q} \left( \int \rho_0^q d\Omega - 1 \right). \quad (17)$$

After the contact and being in thermal equilibrium, new Tsallis' entropy  $S_q$  of the total system is expressed by Eq. (1) with the optimal probability distribution and the generalized canonical partition function, respectively,

$$\rho = \frac{1}{Z_q} [1 - (1-q)\beta^*(H - U_q)]^{-\frac{1}{1-q}}; \quad (18)$$

$$Z_q = \int [1 - (1 - q)\beta^*(H - U_q)]^{\frac{1}{1-q}} d\Omega \tag{19}$$

where  $H$  is the new Hamiltonian of the total system,  $U_q$  is the normalized  $q$ -expectation value of the total energy,  $U_q = \langle H \rangle_q = \int \rho^q H d\Omega / \int \rho^q d\Omega$ , and  $\beta^* = \beta / \int \rho^q d\Omega$  with  $\beta$  the Lagrange multiplier associated with this energy constraint and identified with the inverse temperature  $T$  of the total system. Then, the change of the entropy after and before the contact of the two nonextensive systems can be calculated as

$$S_q - S_{q0} = \frac{k}{1 - q} \int (\rho^q - \rho_0^q) d\Omega = \frac{k}{1 - q} \int \rho^q \left[ 1 - \left( \frac{\rho_0}{\rho} \right)^q \right] d\Omega. \tag{20}$$

Let  $X_q(\rho_0, \rho) = k[1 - (\rho_0/\rho)^q]/(1 - q)$  and  $c_q = \int \rho^q d\Omega$ , Eq. (20) becomes

$$S_q - S_{q0} = c_q \langle X_q(\rho_0, \rho) \rangle_q \tag{21}$$

where we have introduced the so-called  $q$ -expectation value of  $X_q(\rho_0, \rho)$  by  $\langle X_q \rangle_q = \int \rho^q X_q d\Omega / \int \rho^q d\Omega$ . It is further written as

$$\begin{aligned} \langle X_q \rangle_q &= \frac{1}{c_q} \frac{k}{1 - q} \int \rho^q [1 - (\rho_0/\rho)^q] d\Omega = \\ &= \frac{k}{1 - q} \int \rho [1 - (\rho_0/\rho)^q] d\Omega - \frac{1}{c_q} \frac{k}{1 - q} \int (\rho^q - \rho_0^q) (c_q \rho^{1-q} - 1) d\Omega \\ &= \langle X_q \rangle + \frac{k\beta^*}{c_q} \int (\rho^q - \rho_0^q) (H - U_q) d\Omega \end{aligned} \tag{22}$$

where  $\langle X_q \rangle = \int \rho X_q d\Omega$  is the standard expectation value of  $X_q(\rho_0, \rho)$ . With  $(\rho_0/\rho) > 0$ ,  $q > 0$ , we find the inequality,

$$X_q = \frac{k}{1 - q} \left[ 1 - \left( \frac{\rho_0}{\rho} \right)^q \right] \geq \frac{qk}{1 - q} \left[ 1 - \left( \frac{\rho_0}{\rho} \right) \right], \tag{23}$$

and the equality holds if and only if  $\rho = \rho_0$ . Consequently, we have, for  $q > 0$ ,

$$\langle X_q \rangle \geq 0. \tag{24}$$

If the interaction between the two nonextensive subsystems is small so that, after the contact, the Hamiltonian of total systems can be still written as  $H = H_1 + H_2$  and the total energy remains unchanged, namely

$$\frac{\int \rho^q H d\Omega}{\int \rho^q d\Omega} = \frac{\int \rho_0^q H d\Omega}{\int \rho_0^q d\Omega} \tag{25}$$

then the second term on the right-hand side of Eq. (22) vanishes. From Eq. (21) we find that the inequality for Tsallis entropy becomes

$$S_q - S_{q0} \geq 0. \tag{26}$$

This inequality shows that, after the two nonextensive systems are in contact with each other and reach the thermal equilibrium, the total Tsallis entropy cannot decrease, and it is unchanged if and only if  $\rho = \rho_0$ . The two systems being in contact with each other can be treated as an isolated system, in which any processes taking place inside the system can be regarded as adiabatic ones. Thus, we have verified the principle of entropy increase for Tsallis entropy in the above example of nonextensive systems and have made the generalization of the inequality (8) in the framework of NSM.

#### 4. Conclusion

In conclusion, we have studied the property of Tsallis entropy and have examined the principle of entropy increase in the framework of NSM. By considering two nonextensive systems with different temperatures in contact with each other and are in thermal equilibrium, we investigate the change of Tsallis' entropy. We demonstrate that, after the contact, the Tsallis entropy of the total nonextensive system cannot decrease, which leads to a generalization of the principle of entropy increase in the framework of NSM.

*Additional remarks:* The definition of heat in NSM was suggested by Rajagopal (2003) with density matrix in the quantum theory and a similar formulation to the Clausius' inequality was established by Abe & Rajagopal (2003) in the framework of nonextensive quantum thermodynamics.

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